From Box and Diamond: Mini Essays in Honor of Krister Segerberg

WHAT IS SEGERBERG'S TWO-DIMENSIONAL MODAL LOGIC?

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I. Introduction

A characteristic feature of modal logics is that sentences are not taken to be simply true or false, but true or false "at a point". This makes it possible to do justice to the fact that the truth value of a sentence may depend in a uniform way on a piece of information not contained in the sentence. It may depend, for example, on the time the sentence is uttered, the person who utters the sentence, or the time that person is referring to. The truth value of a sentence may also depend, however, on two such pieces of information, so it is appropriate to consider logics in which sentences are taken to be true or false at pairs of points. Segerberg's B is such a logic.

Formulas of B are built up in the usual way from a set of sentence letters using the Boolean connectives and the additional unary connectives □, ⊟, ◫, ⊖, ⦶ and ⊗. A _model_ is a structure (D,V) such that D is a non-empty set and V is a function from the sentence letters to sets of ordered pairs of members of D. If a and b are members of D, _truth_ at (a,b) (in the model (D,V)) is defined by adding these clauses to the usual definition:

$(a,b) \models □A$ iff $\forall x,y \in D \quad (x,y) \models A$
$(a,b) \models ⊟A$ iff $\forall x \in D \quad (x,b) \models A$
$(a,b) \models ◫A$ iff $\forall y \in D \quad (a,y) \models A$
$(a,b) \models ⊖A$ iff $(b,b) \models A$
$(a,b) \models ⦶A$ iff $(a,a) \models A$
$(a,b) \models ⊗A$ iff $(b,a) \models A$

II. Interpretations

Segerberg suggests that □, ⊟, ◫, ⊖, ⦶ and ⊗ be read "everywhere", "everywhere on this latitude", "everywhere on this longitude", "at the diagonal point on this latitude", "at the diagonal point on this longitude" and "at the mirror point", respectively.

But there is another interpretation which is equally illuminating. The formulas of B can be regarded as formulas of predicate logic containing at most two variables, x and y. The sentence letters on this interpretation are predicate letters followed by x and y (in that order). Prefixing the connective ⊟ (◫, □) to a formula is quantifying with respect to x (y, x and y). Prefixing the connective ⊗ is simultaneously replacing all (free and bound) occurrences of x by y and y by x. Prefixing the connective ⊖ (⦶) is the result of changing free x's (y's) into free y's (x's) without introducing new variables. To be precise, call an occurrence of a variable in A <u>x-rated</u> if it is bound by a quantifier whose scope contains an occurrence of x which is free in A. Call an occurrence <u>xx-rated</u> if it is bound by a quantifier whose scope contains an x-rated variable. Then prefixing ⊖ is simultaneously replacing free x's by y, x-rated y's by x and xx-rated x's by y. Prefixing ⦶ is simultaneously replacing free y's by x, y-rated x's by y and yy-rated y's by x.

It is now possible to show that A (regarded as a modal formula) is true at (a,b) iff A (regarded as a formula of predicate logic) is true when a is assigned to x and b is assigned to y. Furthermore, for every formula of dyadic predicate logic with variables from $\{x,y\}$ there is a formula of B which is true under the same conditions. So Segerberg's two-dimensional logic is just the two-variable fragment of dyadic predicate logic.

III. Economies and Generalizations

Not all the connectives of B are needed. ◫ can be defined by ⊗⊟⊗, ⦶ by ⊗⊖, and □ by ⊟⊗⊟⊗. On the other hand, by adding to the stock of connectives we can extend B to an n-dimensional logic B^n equivalent to the n-variable fragment of n-adic predicate logic. An economical way to do this is to introduce the connective Q with the property that $(a_1,\ldots,a_n) \models QA$ iff $(a_2,\ldots,a_n,a_1) \models A$. By suitably combining the sentence letters of all the logics B^n, we obtain a logic of "mixed" dimension which is equivalent to the full predicate calculus. (See Chapter 4 of the author's dissertation for details.)

Segerberg proves that B^2 is decidable by showing that any consistent sentence is true in a finite model. No such proof is possible for B^3--the formula $\forall x\text{-}Fxx \;\&\; \forall x\, \exists y\, Fxyy \;\&\; \forall x \forall y \forall z\, ((Fxyy \;\&\; Fyzz) \rightarrow Fxzz)$ has no finite models. In fact it follows from a result of Kahr, Moore and Wang that for $n > 2$ B^n is undecidable.

Bibliography

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