Y039 -- Draft of 4/15

Modal Logic

Modal logic, narrowly conceived, is the study of principles of reasoning involving necessity and possibility. More broadly, it encompasses a number of inferential systems structurally similar to those that have been devised for necessity and possibility. In this sense, deontic logic (which concerns obligation, permission and related notions), and epistemic logic (which concerns knowledge and related notions) are branches of modal logic. Still more broadly, modal logic is the study of the class of all possible formal systems of this nature.

It is customary to take the language of modal logic to be that obtained by adding one-place operators □ and ◊ for necessity and possibility to the language of classical propositional or predicate logic. Necessity and possibility are interdefinable in the presence of negation: **□A↔¬◊¬A** and **◊A↔¬□¬A** hold. A modal logic is a set of formulas of this language that contains these biconditionals and meets three additional conditions: 1) it contains all instances of theorems of classical logic, 2) it is closed under modus ponens (i.e., if it contains **A** and **A→B** it also contains **B**) and 3) it is closed under substitution (i.e., if it contains **A** then it contains any substitution instance of **A**—any result of uniformly substituting formulas for sentence letters in **A**). To obtain a logic that adequately characterizes metaphysical necessity and possibility requires certain additional axiom and rule schemas:

K **□(A→B)→(□A→□B)**

T **□A→A**

5 **◊A→□◊A**

Necessitation **A/□A**

By adding these and one of the □-◊ biconditionals to a standard axiomatization of classical propositional logic one obtains an axiomatization of the most important modal logic, S5, so named because it is the logic generated by the fifth of the systems in Lewis and Langford's Symbolic Logic. S5 can be characterized more directly by possible worlds models. Each such model specifies a set of possible worlds and assigns truth values to atomic sentences relative to these worlds. Truth values of classical compounds at a world w depend in the usual way on truth values of their components. □**A** is true at w if **A** is true at all worlds of the model; ◇**A**, if **A** is true at some world. S5 comprises the formulas true in all such models. Many modal logics weaker than S5 can be characterized by models which specify, besides a set of possible worlds, a relation of accessibility or relative possibility on this set. □**A** is true at a world w if **A** is true at all worlds accessible from w, i.e., at all worlds that would be possible if w were actual. Of the schemas listed above, only **K** is true in all these models, but each of the others is true when accessibility meets an appropriate constraint.

The addition of modal operators to predicate logic poses additional conceptual and mathematical difficulties. On one conception a model for quantified modal logic specifies, besides a set of worlds, the set D_w of individuals that exist in w, for each world w. ∃**x**□**A**, for example, is true at w if there is some element of D_w that satisfies **A** in every possible world. If **A** is satisfied only by existent individuals in any given world ∃**x**□**A** thus implies that there are necessary individuals—individuals that exist in every possible world. If **A** is satisfied by non-existents there can be models and assignments that satisfy **A**, but not ∃**xA**. Consequently, on this conception modal predicate logic is not an extension of its classical counterpart. Alternative conceptions have more serious disadvantages.

The modern development of modal logic has been criticized on several grounds, and some philosophers have expressed skepticism about the intelligibility of the notion of necessity that it is supposed to describe. (See MODAL LOGIC, PHILOSOPHICAL ISSUES IN.)

6 General results

1 History

Modal logic in the narrow sense was a topic of considerable interest to ancient and medieval philosophers. It occupied two chapters of Aristotle's De Interpretatione, and a substantial part of the Prior Analytics. Discussion of argument forms involving necessity and possibility that included, and sometimes transcended, commentary on Aristotle's was standard fare in Hellenistic and Medieval treatises on logic (see MEDIEVAL LOGIC). From our vantage-point the ancient and medieval discussion can be interpreted as including distinctions among various kinds of possibility and necessity and investigations of the logical relations among them as well as logical investigations of the interactions between modalities and negation, modalities and conditionals or consequence, and modalities and quantifier expressions. Aristotle determines in De Interpretatione, for example, that *it may be* and *it cannot be* are contradictories, as are *it may not be* and *it cannot not be*. Furthermore, 'from the proposition *it may be* it follows that it is not impossible' and in one sense 'the proposition *it may be* follows from the proposition *it is necessary that it should be*'. In another sense (which we might gloss as *it is merely possible that*), *it may be* is logically incompatible with *it is necessary that it should be*.

Besides these purely modal principles, Aristotle and his commentators were concerned with arguments that we might think of as mixing time and modality. A notorious example is the fallacious 'sea battle' argument for determinism that he tries to debunk in De Interpretatione. (See DETERMINISM.) In addition to the admixture of temporal considerations, one should observe that the notion of necessity involved in these discussions is not likely to be the same as the one whose logical behavior was summarized above. Aristotle himself catalogs four senses of the word *necessary* in Metaphysics V Ch.5, and makes other distinctions elsewhere.

2.

Although necessity and possibility have never ceased playing an important role in philosophical discourse, their logical properties were largely neglected in modern philosophy until the beginning of this century. The contemporary revival was sparked by C.I. Lewis's critique of Russell and Whitehead's Principia Mathematica. The logical system elaborated in Principia contained as theorems the formulas $\mathbf{p{\supset}(q{\supset}p)}$ and $\sim\mathbf{p{\supset}(p{\supset}q)}$, which Whitehead and Russell understood as asserting the apparently paradoxical propositions that if a sentence is true it is implied by any sentence and if a sentence is false it implies any sentence. Lewis maintained that these propositions—while unavoidable and unobjectionable with respect to Russell and Whitehead's understanding of implication—were false with respect to a more natural 'strict' sense of implication. He embarked on a project of determining the appropriate axioms of strict implication with which to supplement the Principia system. In Principia the 'material' implication $\mathbf{p{\supset}q}$ is considered true unless $\mathbf{p}$ is true and $\mathbf{q}$ false. In Lewis's systems, the strict implication $\mathbf{p{\Rightarrow}q}$ is considered true only if it is [[impossible]] that $\mathbf{p}$ is true and $\mathbf{q}$ is false. Thus Lewis's strict implication can be defined from Russell and Whitehead's $\supset$ (and the negation sign, $\sim$) and a new connective, $\Diamond$, of possibility: $\mathbf{p{\Rightarrow}q} = \sim\Diamond\mathbf{(p\&{\sim}q)}$. Conversely, possibility can be defined from strict implication: $\Diamond\mathbf{p} = \sim\mathbf{(p{\Rightarrow}{\sim}p)}$. Hence Lewis's project of finding the correct logical principles for his notion of strict implication is tantamount to that of finding the correct logical principles for possibility or, equivalently, those for necessity. Lewis and Langford's Symbolic Logic describes five different axiom systems as candidates for the logic of strict implication. Much effort was expended in the first half of the century investigating these systems and variations of them. Even showing that all five are distinct (in the sense that they produce different classes of theorems) required considerable ingenuity. Fifteen years after the publication of Symbolic Logic, Carnap gave a non-axiomatic characterization of 'logical' necessity (see Carnap 1947). <Necessarily A> is true, according to Carnap, if **A** is 'L-true', i.e., if **A** is true in all state descriptions. (A state description is a kind of canonical inventory of the primitive relations that hold and fail to hold of each sequence of individuals). Thus Carnap can be seen as making precise the old idea that necessity is truth in all possible worlds. This idea is usually associated with Leibniz, but it is traced to Descartes by Curley (see Curley 1984) and to Duns Scotus by Knuuttila (see Knuuttila 1982). The logic determined by Carnap's interpretation turned out to be S5, the fifth of the Lewis and Langford systems. In the late fifties and early sixties, several authors proposed interpretations that refined and generalized Carnap's idea by introducing something like the accessibility relation described above (See Kanger 1957, Montague 1960, Hintikka 1963, Kripke 1963). Kripke models, which are essentially the models described above, are the neatest formulation of this idea. Kripke and a continuing line of successors have shown that a great variety of modal systems can be characterized by models of this kind. This enormously simplified the kinds of investigations of axiomatic systems mentioned above and opened new lines of research.

Kripke models were particularly fruitful in the study of modal logic in the broader senses. It had long been noted that the pairs *it will always/sometimes be the case that, it is obligatory/permitted that* and *it is known/consistent with knowledge that* exhibit logical behavior similar to that of *necessarily/possibly*. The success of Kripke's treatment of necessity encouraged analogous treatments of these other notions. **GA** (<it will always be the case that **A**>) is true at time t if **A** is true at all times after t; **FA** (<it will sometimes be the case that **A**>) is true at t if **A** is true at some times after t. **OA** (<it is obligatory that **A**>) is true at world w if **A** is true at all worlds at which the obligations of w are discharged; **PA** (<it is permitted that **A**>) is true at w if **A** is true at some such worlds. **KA** (<it is known that **A**> is true at world w if **A** is true in all worlds consistent with what is known at w. The resulting systems are labeled tense logic, deontic logic, and epistemic logic to distinguish them from the original alethic modal systems for necessity. In the last two areas this account probably takes the analogy with necessity too far, but it still serves as a useful point of departure. (See DEONTIC LOGIC and EPISTEMIC LOGIC.) Among other broadly modal systems that have received attention recently are the dynamic logics or logics of computation for reasoning about computer programs. Here worlds become computation states, which can be thought of as specifications of all the program variables at a particular time, and relative possibility becomes program accessibility, which holds between two states if a program can start in the first and terminate in the second. Such logics may be useful in verifying, without interminable testing, that a large program is |correct|, i.e., that it does what it is supposed to do. (See DYNAMIC LOGIC.)

The development of accessibility semantics led naturally to general questions about the classes of systems that can be characterized by various versions of it. Some of these are discussed below. Before the development of the accessibility interpretation, questions about the scope of axiomatic systems were often answered by devising suitable algebraic interpretations, and algebraic methods remain important tools for studying more general questions. Possible worlds semantics, however, seems less ad hoc than the algebraic. It is not clear whether they provide an analysis of necessity and possibility, or whether the notions that they incorporate—possible world and accessibility—are themselves to be analyzed in terms of necessity and possibility. Either way, there is a close fit between the meanings of modal terms and their possible worlds interpretations. The notion of possible world has proved useful in philosophical discussions on topics other than necessity and possibility—supervenience, causality, and the nature of propositions, properties and relations, to mention a few examples. The general utility of possible worlds has, in turn, inspired modal languages more expressive than the standard box-and-diamond variety.

3. The System S5

As suggested above, S5 is the set of formulas provable from the axioms of classical propositional logic (henceforth PL), the axiom schemas **K**,**T**,**5**, and $\Diamond$**A**↔¬□¬**A** (henceforth **Df**$\Diamond$) and the rules modus ponens (henceforth MP) and necessitation (henceforth Nec). It is important to understand that Nec states □**A** is a theorem if **A** is, and not that □**A** is true if **A** is. Given **T** and the replacement of equivalents, the latter condition would make □**A** and **A** freely interchangeable, rendering modal logic pointless.

To identify a modal system in this way with its theorems and to refer to such systems as modal logics is to follow standard, though misleading, practices. One would expect a |logic| to indicate which conclusions follow logically from which premises, and perhaps the deductions by which they so follow. Furthermore, although S5 and other modal systems are intended to generate formulas true in virtue of form, it is not clear that they are intended to generate formulas true in virtue of logical form, for it is not clear that necessity is a logical particle (see LOGICAL FORM and LOGICAL CONSTANT). The first of these concerns can be reduced by stipulating that **A** follows from a set of formulas with respect to a logic L, if and only if L contains a conditional whose antecedent is a conjunction of those formulas and whose consequent is A.

A model (for S5) is a pair (W,V) where W is a non-empty set (the possible worlds) and V is a function (the valuation function) that assigns a truth value (**T** or **F**) to each sentence letter and each possible world $w \in W$. If M=(W,V) is a model and $w \in W$, the notion A is true at w in M (written $(M,w) \models$ **A**) is defined inductively, the key clauses being: $(M,w) \models \Box$**B** if, for all $v \in W$, $(M,v) \models$ **B**; and $(M,w) \models \Diamond$**B** if, for some $v \in W$ $(M,v) \models$ **B**. A formula is true in M if it is true at all possible worlds in M, and it is valid, if it is true in all S5-models.

The soundness theorem for S5 says that the semantics respects the logic in the sense that every formula in the logic is valid. The sufficiency theorem states that every valid formula is in the logic. To say that S5 is complete with respect to the interpretation given is to say that it is both sound and sufficient, i.e., that validity and theoremhood coincide. Soundness is proved by an easy inductive argument appealing to the axiomatization: Each axiom is observed to be valid and the two rules are shown to preserve validity. Sufficiency requires more ingenuity. A common approach is to adapt Henkin's proof of the completeness of classical logic. (For details see Lemmon 1966 or Chellas 1980.) One can get an idea of the value of an interpretation and completeness theorem by trying, with and without them, to demonstrate that $\Box \mathbf{p} \rightarrow \Box\Box \mathbf{p}$ is a theorem of S5.

In S5, all |nesting| or |iteration| of modality can be eliminated. For example, if a formula has the form $M_1...M_nA$ where each M_i is either □, ◊, or negation, an equivalent formula can be obtained by deleting all the □'s and ◊'s except the innermost. Taking a modality to be a string of □'s, ◊'s and ¬'s, it follows that there are only six non-equivalent modalities in S5: the empty string, ¬,□,◊,¬□ and ¬◊, corresponding to simple truth, falsity, necessity, possibility, non-necessity and impossibility.

One important schema of classical propositional logic that is [[not]] provable in S5, is

(**Ext**) **(A↔A')→(B↔B')** where **B'** is the result of replacing an occurrence of the subformula **A** in **B** by **A'**.

For example, if **p** and **q** are both true at world w but only **q** is true at world v, then **(p↔q)→(□p↔□q)** is false at w. By completeness it follows that **(p↔q)→(□p↔□q)** is not a theorem of S5. **Ext** (extensionality) says that replacement of one subformula by another of the same truth value will not affect the truth value of the whole. Its failure is often viewed as characteristic of modal systems in general. Note however, that provably equivalent formulas can be substituted for each other.

S5 is a relatively strong modal system. Its only extensions are the trivial logic, containing all instances of **□A↔A** and, for every natural number n, the n-possibility logic, containing all instances of $(\Diamond A_1 \wedge ... \wedge \Diamond A_{n+1}) \rightarrow Dis$ where **Dis** is the disjunction of all formulas $\Diamond(A_i \wedge A_j)$ such that $i<j\leq n+1$. (n-possibility logic is complete with respect to the class of all models with at most n possible worlds.)

The formulas of (propositional) S5 correspond to formulas of classical monadic predicate logic in one variable. $\Diamond(p_1 \wedge \Box p_2)$ for example, corresponds to $\exists x(P_1x \wedge \forall x P_2x)$. Since decision procedures for monadic predicate logic are known, this correspondence allows one to effectively determine whether a formula is in S5 by testing the corresponding monadic formula for validity. A more direct proof of decidability rests on the result that S5 has the finite model property: every non-theorem is false in some model with finitely many worlds. To test whether A is in S5, one checks first whether A is provable in one step or falsifiable in a one-world model, then whether it is provable in two steps or falsifiable in a two-world model, and so on. Each step can be completed in finite time. By the finite model property, there is some step n at which the process yields the desired answer.

The most important philosophical question about S5 is whether it captures the inferences and truths it was intended to. This may depend, of course, on the kind of necessity that □ is supposed to represent. On the usual view this is broadly logical or metaphysical necessity. Truths necessary in this sense include those true in virtue of logical form (the logical truths), those true in virtue of meaning (the analytic truths) and a more problematic category of those true in virtue of the basic nature of things. The last category has been said to contain truths of mathematics, the proposition that water is H_2O , the proposition that Queen Elizabeth came from the egg and sperm that she did (see ESSENTIALISM). All these examples are controversial, but those who argue whether particular propositions are metaphysically necessary may nevertheless share a common conception of what it is to be metaphysically necessary. Furthermore, examples of propositions that lack metaphysical necessity seem uncontroversial: *Napoleon invaded Russia*, *Asbestos is carcinogenic*, *Paris is the capital of France*. The question of whether S5 (or any other system) is the right |logic| for metaphysical necessity can be divided into two parts, corresponding to the two parts of the completeness theorem. Say that S5 is correct if every theorem represents a formal truth about metaphysical necessity. Say that it is adequate if every formal truth (with *and*, *not*, *or* and *if* as well as *necessarily* and *possibly*) is represented by a theorem. Correctness and adequacy, then, are philosophical counterparts to soundness and sufficiency.

Correctness can be established by an argument similar to that for soundness: first show that the axioms represent formal truths and then that the rules transform formulas representing formal truths into formulas representing formal truths. Axioms **T**, for example, clearly represent formal truths. (*If necessarily 87+25=112 then 87+25=112* is true in virtue of its form). The difficult cases are axioms **5** and rule Nec. For the former case we need to establish that sentences of the form <If possibly S then necessarily possibly S> are true in virtue of their form. One argument is as follows. To say possibly S is to say that <not S> doesn't follow logically or analytically from a description of the natures of things or from logical truths or analytic truths, i.e., that S is logically and analytically consistent with the basic natures of things. But a proposition that something is consistent with logical laws, meanings, and basic natures in this way is true or false in virtue of those logical laws, meanings, and basic natures. So <possibly S>, if true, is necessarily true. And since this argument does not appeal to S, the conditional is true in virtue of its form. An argument for the rule Nec, might go as follows. Once it is established that all the axioms represent formal truths and that MP preserves formal truth, we know that everything proved without Nec represents a formal truth. Since formal truths are either logical or analytic, they are necessarily true. Since the argument that any of these sentences S is necessary relies only on the form of S, necessarily S is true in virtue of its form. This establishes that the first application of Nec results in formulas that represent formal truths. But if a sentence S proved with one application of Nec is shown to be a formal, and hence necessary, truth by an argument that appeals only to S's form, then <necessarily S> must be true in virtue of its form. In this way, the argument can be extended to any subsequent application of Nec.

Adequacy might be established indirectly. Suppose that S5 were not adequate for metaphysical necessity. Then there would be some formal truth S with *necessarily*, *possibly*, *and*, etc., represented by a formula **A** that is not a theorem of S5. We should, then, be able to |improve| the adequacy of S5 by adding **A** as an axiom. Furthermore, all the substitution instances of **A** will represent sentences with the form of S, which, if S is a formal truth, will also be formal truths. So we should be able to add all the substitution instances of **A** as well, obtaining an extension of S5. But, as was noted above, the extensions of S5 must contain either $\Box\mathbf{A}\leftrightarrow\mathbf{A}$ or, for some n, $(\Diamond \mathbf{A}_1 \wedge ... \wedge \Diamond \mathbf{A}_{n+1}) \rightarrow \mathbf{Dis}$. All these are incorrect for metaphysical necessity.

4. Quantified S5

Consider a language obtained by adding □ and ◊ and a special predicate **E** of existence to a version of predicate logic with predicates P_1, P_2, ... , individual constants, t_1,t_2,..., and the identity sign, =. A model is a triple (W,D,V) where W is a non-empty set (the possible worlds), D is a function that assigns to each w∈W a set D_w (the domain of w) and V is a function that assigns to each constant t a member of the union ∪D of the sets D_w (the possible object denoted by t_i) and, to each world and n-place predicate, an n-ary relation on ∪D. A definition of truth at a world is obtained from the previous one by replacing the base clause and adding clauses for ∀, ∃, =, and **E**. The quantifier clauses state that, for example, ∀**xPx** is true at w if, for every d in D_w, **Pt** is true at w when **t** denotes **d**, and that ∃**xPx** is true at w if, for some d in D_w, **Pt** is true at w when **t** denotes d. **s=t** is true at w if **s** and **t** denote the same possible objects. **Es** is true at w if V(**s**)∈D_w.

This interpretation reflects several choices. First, constants are objectual, i.e., V assigns a possible object directly to each constant. □**Pcd** is true at w if the possible objects V(**c**) and V(**d**) are related, at every world w, by V(P,w). Thus □**Pcd** expresses a de re necessity. It asserts that particular objects, independently of their descriptions, are necessarily related. This treatment of constants makes the schemas **(c=d)**→□**(c=d)** (necessary identity) and ¬**(c=d)**→□¬**(c=d)** (necessary difference) valid. Kripke has influentially argued that this is appropriate if **c** and **d** represent proper names of natural language, and inappropriate if they represent definite descriptions (see PROPER NAMES). The alternative to treating constants objectually is to allow their denotations to vary from world to world. An (individual) concept is a function from worlds to individuals. For example, the concept *first person to reach the South Pole* might assign Amundsen to this world, Scott to the possible world in which Scott wins his race with Amundsen, and nothing to possible worlds in which the Pole is never (or always) occupied by people. We can regard a constant with a non-objectual interpretation as denoting a concept rather than an individual. On this |conceptual| interpretation, necessary identity and necessary difference are not valid. Quantification may be treated conceptually as well: ∀**xA** is declared true at w if every individual concept assigns to w an object to which **A** applies at w. In that case, quantified necessary identity and difference formulas—$\forall$**x**$\forall$**y(x=y**→□**x=y**) and $\forall$**x**$\forall$**y(¬x=y**→□**¬x=y)**—lose their validity as well. The quantification described above, by contrast, is objectual.

Second, quantification is actualist. The truth of $\forall$**xPx** at w requires only that the objects in D_w have the appropriate property. It follows that the Barcan formula, $\forall$**x**□**Px**→□$\forall$**xPx**, and its converse, □$\forall$**xPx**→$\forall$**x**□**Px**, are both invalid. Suppose, for example, that for some world w, **P** is a predicate that holds everywhere of just the objects that exist in w, and that D_u contains something not in D_w. Then at w, *everything is necessarily **P*** is true while *necessarily, everything is **P*** is false. Conversely suppose that, at every world w, **P** holds of just the objects that exist at w, and that D_v contains something not in D_u. Then at v *Necessarily everything is **P*** is true, while *Everything is necessarily **P*** is false. More generally, $\forall$**x**□$\forall$**y**□**A**, □$\forall$**x**$\forall$**y**□**A**, □$\forall$**x**□$\forall$**yA**, and □$\forall$**x**□$\forall$**y**□**A** are logically distinct. If quantification is possibilist, i.e., if $\forall$**xA** means that **A** is true of all objects in $\cup$D, then these formulas are all equivalent and the Barcan formula and its converse are valid. The distinction between actualist and possibilist quantification is significant only because the models defined above have domains that vary from world to world. If one stipulates constant domains, i.e., that D_u=D_v for all worlds u and v, then the possible objects are just the actual ones, and the distinction collapses. Barcan and its converse are again valid.

Third, predicates can be truthfully applied to constants denoting non-actual objects. Since quantification is actualist, this implies that the classical theorems **Pc**→$\exists$**xPx** and $\forall$**xPx**→**Pc**, are not valid. These principles can be retained by insisting that the application of predicates to constants denoting non-actuals is always false (defining models so that V($\mathbf{P_i}$,w) is a relation on D_w). But to do so would be to adopt a kind of atomism according to which the properties and relations expressed by atomic formulas with free variables have a special status. Furthermore, although it would save the particular classical theorems above, some of their substitution instances—like **¬Ex**→$\exists$**x¬Ex**—would still fail. Another alternative is to stipulate that formulas in which predicates are applied to constants denoting non-existents lack truth value. If validity is taken as 'false in no model', both the formulas and their instances are saved. On the other hand that approach would seem to make $\Box$**Ec** valid (if necessity is 'nowhere false') or to make $\Box$**(Fc**$\vee\neg$**Fc)** invalid (if necessity is 'everywhere true').

Finally, domains are permitted to be empty. This implies that $\Diamond\exists$**x(Fx**$\vee\neg$**Fx)**, $\Box\exists$**x(Fx**$\vee\neg$**Fx)** and $\exists$**x(Fx**$\vee\neg$**Fx)** are all invalid. If one regards the first as expressing a formal truth, one can require that some world have non-empty domain; if the second, that all worlds do. If one regards the third as expressing a formal truth one can require that each model specify—in addition to the possible worlds, domains, and valuation—a particular possible world (the actual) that has a non-empty domain. Truth in a model is then redefined as truth at the actual world of the model

5. Weaker systems

A Kripke model is a triple (W,R,V) where W and V are as in §3, and R (accessibility) is a binary relation on W. Truth at a world is defined as before except for the $\Box$ and $\Diamond$ clauses: (M,w)$\models\Box$**A** if, for all v$\in$W such that wRv, (M,v)$\models$**A**; (M,w)$\models\Diamond$**A** if, for some v$\in$W such that wRv, (M,v)$\models$**A**. Truth in a model and validity are defined as before. The formulas valid in this sense comprise the logic K. K can be axiomatized by the schemas **PL**, **K**, **Df**$\Diamond$ and the rules MP and Nec. Since it lacks the schema $\Box$**A**$\rightarrow$**A**, K is not adequate for necessity under any construal. It occupies an important position in modal logic in the broader sense, however, because many well-known modal systems are simple extensions of it. The systems in the leftmost column of the table below, for example, are obtained by adding the schemas in the middle column to **PL**, **K** and **Df**$\Diamond$ and the rules MP and Nec.

System	Characteristic Axioms	Conditions on R
D	**D**: □**A**→◊**A**	seriality: ∀x∃yRxy
T	**T**	reflexivity: ∀xRxx
S4	**T** **4**: □**A**→□□**A**	reflexivity transitivity: ∀x∀y∀z(Rxy&Ryz→Rxz)
S4.3	**T**,**4** **H**: **◊A&◊B→◊(A∧◊B)∨◊(◊A∧B)∨◊(A∧B)**	reflexivity, transitivity connectedness: ∀x∀y(Rxy∨Ryx)
GL	**W**: **□(□A→A)→□A**	transitivity no infinite chains: $Rx_1x_2\&Rx_2x_3\&\ldots\rightarrow\exists i(x_i=x_{i+1})$

The schema **D** is formally true when □ and ◊ are read *it is obligatory that* and *it is permitted that*, and the system **D** is known as the standard deontic logic. T was one of the earliest modal logics to be characterized precisely (see Feys 1937, 1938) and it seems to be the weakest logic in which □ can be plausibly regarded as representing a reading of *it is necessary that*. If, as some have suggested, the remaining S5 axioms are not correct for physical necessity, T would be a plausible candidate for the logic of that notion. S4 was the fourth of the Lewis systems. Gödel gave it a characterization like the one above and showed it to be intertranslatable with intuitionistic propositional logic (see Gödel 1933) . S4.3 (so named, in part, because it is intermediate in strength between S4 and S5) is correct and adequate for a reading of □ as *it is and always will be the case that* if time is assumed to be linear (see TENSE AND TEMPORAL LOGIC). If sentence letters represent statements about numbers and □ is interpreted as *provable in arithmetic* then the system GL contains exactly the formulas that are themselves provable in arithmetic (where a statement about provability is |provable| if its Gödel number is the number of a theorem of arithmetic). By adding all instances of $\Box\mathbf{A}\rightarrow\mathbf{A}$ to the theorems of GL (without allowing any new applications of the rule Nec) one obtains the system GLS, which may be regarded as the logic of arithmetic provability. Whereas GL comprises the arithmetically provable formulas about provability, GLS comprises the truths about provability (see PROVABILITY LOGIC).

Each of these logics can also be characterized semantically, just as K was. A D-model is a Kripke model whose accessibility relation is serial, i.e., every world is related to some world or other, and models appropriate for each of the other systems can be similarly defined by the conditions in the table above. In each case, soundness and completeness results like the one for S5 sketched above can be given.

Not all broadly |modal| systems can plausibly be interpreted by Kripke models. Suppose $\Box\mathbf{A}$ is read *usually **A***. There is no relation on times such that *usually **A*** is true now if **A** is true at all related times. Rather, the truth of *usually **A*** depends on the number, and perhaps the distribution, of all the times at which **A** is true. This suggests a more general kind of modal semantics, one formulation of which is the neighborhood semantics. A neighborhood model is a triple (W,R,V) where W and V are as before, and R is a relation between worlds and sets of worlds (the neighborhoods of those worlds, although there is no requirement that the neighborhood of a world contain the world itself, or even that it be non-empty). The definition of truth at a world is as before except that $\Box\mathbf{A}$ is true at w just in case w is related to the truth-set of **A**, i.e., the set of worlds at which **A** is true; and $\Diamond\mathbf{A}$ is true at w if w is unrelated to the |falsity-set| of **A**. The modal system that is determined by the set of all such models is the system E, axiomatized by the schemas **PL** and **Df**$\Diamond$ and rules MP and equivalents (RE), 'if $\mathbf{A}\leftrightarrow\mathbf{B}$ is provable, so is $\Box\mathbf{A}\leftrightarrow\Box\mathbf{B}$'. Like K, E provides a convenient base from which a variety of systems of interest can be constructed, rather than a characterization of the formal truths for some particular reading of $\Box$. This idea can be carried even further. OL (Operator logic) is the system in the language with □ that is axiomatized just by PL (see Kuhn 1981). The modal operators of S5, K, E, and OL, can be regarded as being successively more schematic and having successively less content, the theorems of OL being the modal sentences that are true in virtue of their [[logical]] form, and the stronger modal systems being theories of operators based on that logic.

6. General results

Much of the contemporary study of modal logic is directed, not towards investigating any particular one of the systems described above in more detail, but towards a general understanding of classes of such systems. The modal logics, as defined above, form a lattice structure: each pair has a unique minimal extension and a unique maximal sublogic. A Kripke model M=(W,R,V) can be viewed as the addition of a valuation V to the Kripke frame (W,R). M is then said to be based on (W,R). Similarly, the neighborhood model (W,N,V) is based on the neighborhood frame (W,N). A formula is valid in a frame if it is true in all models based on the frame, and it is valid in a class of frames if it is valid in all frames of the class. A frame for L is a frame in which all the theorems of L are valid. The set of formulas valid in a class of frames is a logic, the logic determined by that class. A logic is sound for a class of frames if every formula in the logic is valid in the class; it is sufficient for the class if every formula valid in the class is a member of the logic; it is complete (for the class) if it is sound and sufficient.

The logic determined by a class of frames is, by definition, complete for that class. On the other hand it might be complete for other classes as well. K, for example, is determined by (and therefore complete for) the irreflexive Kripke frames as well as all Kripke frames. In 1974, Fine and Thomason independently showed that there are finitely axiomatizable extensions of K that are not determined by any class of Kripke frames (see Fine 1974 and Thomason 1974). Such logics are incomplete: formulas true in every frame that verifies the axioms are unprovable. It is now known that the incompleteness phenomenon is widespread. For every extension L of the logic T there are uncountably many incomplete logics whose frames are exactly the frames for L.

(This result holds for both Kripke frames and neighborhood frames. See Benton 1985.) The incomplete logics that have been exhibited in the literature are generally complex and ad hoc, but one simple example is obtained from GL by replacing the first conditional of the schema **W** by a biconditional. In the other direction, results of Bull and Fine imply that every extension of S4.3 that admits the necessitation rule is complete and decidable (see Bull 1966 and Fine 1971).

Much work has also been done on the correspondences between modal and classical formulas. The schema **T** corresponds to the classical formula ∀**xRxx** in the sense that the frames for the former are just the first order models for the latter. Most modal schemas that have arisen naturally in philosophical discussion correspond similarly to first order formulas. The schemas in the second column of the table above, for example, correspond to formulas in the third column. The McKinsey schema, □◊**A**→◊□**A**, on the other hand corresponds to no first order formula (see van Benthem 1975 or Goldblatt 1975), and no modal schema corresponds to irreflexivity: ∀**x**¬**Rxx**. (The latter fact follows from the remark above that K itself is complete for the irreflexive frames.)

The general study of modal logic encompasses a variety of other topics, including: model theory (transformations of frames and models may preserve truth values of classes of formulas), boundary investigations (some logics have properties that all their extensions lack), expressive power (some classes of modal connectives can be defined from a few representatives), and connections with non-modal logics (like that between S4 and intuitionistic logic). The completeness and correspondence investigations discussed above, however, have formed the core of contemporary investigations.