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Steven T. Kuhn

Quantifiers as Modal Operators*

Abstract. Montague, Prior, von Wright and others drew attention to resemblances between modal operators and quantifiers. In this paper we show that classical quantifiers can, in fact, be regarded as "*S5*-like" operators in a purely propositional modal logic. This logic is axiomatized and some interesting fragments of it are investigated.

Resemblances between quantifiers and modal operators were noticed very early in the modern development of modal logic. Much of the early work in the field exploits these resemblances, adapting familiar ideas of predicate logic to the unfamiliar modal systems. In this paper our aim will be the reverse. We will show that predicate logic can be regarded as a special kind of propositional modal logic.

The fact that this can be done supports the claims of von Wright and others that quantification is a kind of modality. These claims bear on recent discussions of the value of modal logic. Champions of classical predicate logic have argued that without quantifiers modal logic is trivial and with them it is unintelligible. This paper meets the first half of the criticism by showing that propositional modal logic, broadly construed, is no more trivial than first order logic itself.

In addition it is hoped that a modal reconstruction of predicate logic might provide some insight into predicate logic itself. In our system, for example, quantifiers and variables are replaced by four different operators. We try to take advantage of this "division of labor" to analyze the undecidability of predicate logic.

The paper is organized as follows. Section One surveys two earlier attempts to link quantifiers and modal operators — the first by Richard Montague in [14]; the second, by Arthur Prior in [16] and [17]. In Section Two a system of propositional modal logic called PREDBOX is described and shown to be equivalent to first order predicate calculus. In Section Three PREDBOX is axiomatized and a completeness proof is sketched. The paper concludes with a discussion of the expressive power and decidability properties of some interesting fragments of PREDBOX.

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* This paper is based in part on material in Chapter IV of [9]. I have benefited greatly from comments of Anthony M. Ungar.

I. Background

Montague

Richard Montague, like several authors before him, noticed that many standard modal logics the necessity and possibility operators behave like universal and existential quantifiers. For example, the following properties of ***S5*** all seem to be reasonable conditions on a logic of necessity:[1]

1) If $\vdash A$ then $\vdash \Box A$
2) $\vdash \Box(A \to B) \to (\Box A \to \Box B)$
3) $\vdash -\Box A \leftrightarrow \Box - \Box A$
4) $\vdash \Box A \leftrightarrow \Box \Box A$
5) $\vdash -A \leftrightarrow - \Box - A$

Each of these is mirrored by a law of quantification theory.

1′) If $\vdash A$ then $\vdash \forall x A$
2′) $\vdash \forall x(A \to B)\ (\forall x A \to \forall x B)$
3′) $\vdash -\forall x A\ \forall x - \forall x A$
4′) $\vdash \forall x A \leftrightarrow \forall x\ \forall x A$
5′) $\vdash \exists x A \leftrightarrow - \forall x - A$

This observation led Montague to suggest the following uniform treatment of necessity and quantification.

Consider a language with predicates, individual constants, individual variables, quantifiers over individuals and the unary sentential operator $\Box$. Let a *model* for such a language be a triple $(D, \boldsymbol{R}, f)$ where $(D, \boldsymbol{R}) = (D, R_1, R_2, \ldots, c_1, c_2, \ldots)$ is an ordinary model for first order logic and f is an assignment of members of the domain D to the individual variables. Let two models $(D, \boldsymbol{R}, f)$ and $(D', \boldsymbol{R}', f')$ be related by P if $D = D'$ and $f = f'$. For any individual variable x, let them be related by P_x if $D = D'$, $\boldsymbol{R} = \boldsymbol{R}'$ and f agrees with f' on all variables with the possible exception of x. Then we can write the truth conditions for $\Box$ and $\forall x$ in such a way that the similarity between them surfaces:

$$M \models \Box A \quad \text{iff for all models } M' \text{ such that } MPM', \quad M' \models A.$$

$$M \models \forall x A \quad \text{iff for all models } M' \text{ such that } MP_xM', \quad M' \models A.$$

Thus the addition of quantifiers to a language appears to be no different than the addition of a collection of modal operators.

In light of the subsequent development of semantics for modal logic it would be natural to recast Montague's formulation in terms of relations between possible worlds, rather than relations between models. One way

[1] The only ***S5*** axiom missing is $\Box A \to A$. Montague felt that if $\Box$ represented physical necessity then this axiom, though true, should not be considered a truth of logic.

to do this would be to let Montague's models *be* the possible worlds. A *frame* on this view would be a structure $(W, Px_1, Px_2, \ldots)$ where W is a set of models and Px_1, Px_2 are relations on W like the ones described above. This won't quite do, however, because in predicate logic, unlike propositional logic, the truth values of atomic formulas under an interpretation are not independent of one another. We would like to say that $Px_1 \ldots x_n$ is true at $w = (D, \boldsymbol{R}, f)$ if and only if $(f(x_1), \ldots, f(x_n))$ is a member of the relation which interprets P. But this kind of truth condition doesn't make predicate logic seem very much like modal logic.

Prior

Arthur Prior points out that modal logics deal with sentences which are *open* in the sense that contextual information is needed to assess their truth. A sentence isn't just 'true'; it is 'true-at-a-world' or 'true-at-a-time'. Sentences containing the pronoun 'I' are also open in this sense. Prior suggests a new kind of propositional modal logic, 'egocentric logic' whose sentences are to be evaluated at people. 'I am eating a liverwurst sandwich' is false at the author, though it is probably true at someone. Stretching this notion slightly, we can imagine a logic whose sentences are true or false at objects in general. 'I am inanimate', for example, would be true at the Washington Monument. In the standard modal logics we prefix a box to a sentence and call it *necessarily* true if it is true in all possible worlds. Similarly we could prefix a 'U' to a sentence of egocentric logic and call it *universally* true if it is true at all objects. We would then have a modal version of monadic predicate logic.[2] But there is no need to stop here. The "objects" at which sentences are true or false might be ordered pairs. 'My first member is the author of my second' would be true at (Scott, *Waverly*) but false at (Leibniz, *Critique of Pure Reason*). And if we allow sentences which are evaluated at pairs of objects we might as well allow those which are evaluated at triples and quadruples as well. More generally we might allow sentences to be evaluated at infinite sequences of objects, or *assignments*.

The writings of both Montague and Prior suggest that quantifiers can be construed as operators in a purely propositional modal logic. On Montague's approach the "possible worlds" at which sentences are evaluated are ordinary first order models. On Prior's approach they are sequences of individuals. Either way, however, there is a problem. The atomic formulas of predicate logic cannot all be treated as atoms in the modal language. If we regard Pxy and Pyx, for example, as distinct sentence letters of the modal language then $\exists x \exists y\, Pxy \;\&\; -\exists x \exists y\, Pyx$ will be satisfiable. If we regard them as identical sentence letters then $\exists x\ \exists y\, (Pyx \;\&\; -Pxy)$ will be unsatisfiable.

[2] This was pointed out by David Lewis in [12], p. 112.

A solution to this problem can be found in the variable-free formulations of logic by Tarski, Bernays, Halmos, Nolin and Quine.[3] Each of these authors has devised a set of operations by which we can get the effect of arbitrary permutations and identifications of the variables occurring in a formula. In our case we can express these operations with additional sentential operators. When added to the modal language these operators enable us to represent conveniently all the atomic formulas of predicate logic. The idea is to restrict attention to n-place atomic formulas in which the first n variables occur exactly once and in a fixed order. These can be treated as atoms, and from them, using the sentential operators mentioned above, the remaining "atomic formulas" of PRED can be generated. But since the variables following predicate letters are now uniquely determined; one can just as well dispense with variables altogether.

II. PRED and PREDBOX

In this section we describe a propositional modal system called PREDBOX and show it equivalent to classical predicate logic, hereafter referred to as PRED. The language of PREDBOX differs from the usual modal languages in that its sentences come in *sorts*. Sort-n sentences correspond to first order formulas whose variables are among $v_1, \ldots, v_n$. A sort-n sentence is assigned a truth value only at sequences of length n or greater. I have argued in [9] that languages which are *propositionally many sorted* in this way (i.e., languages in which different kinds of sentences are evaluated at different kinds of indices) arise naturally in many applications of modal logic. Without the sorts, however, it would still be possible to construct a logic equivalent to a kind of predicate logic in which the predicates have no fixed degree. (Such a logic has been investigated by Martin Davis in [2].)

Sentences of PREDBOX are built up from sentence letters using the usual Boolean connectives and the following additional one-place operators:

$\Box$ ("generalization")
ϱ ("rhotation")
σ ("switch")
$[|]$ ("identification")

More formally, for each natural number n, the sort-n formulas of PREDBOX are defined by the following clauses:[4]

[3] See [7], [3], [5], [15] and [18]. One work borrows from these authors, particularly Quine.

[4] In this paper we take *natural numbers* to be positive integers.

i) For every natural number i, p_i^n is a sort-n sentence.
ii) If A is a sort-n sentence, so are $\Box A$, ϱA, σA, $[\|]A$ and $-A$.
iii) If A is a sort-n sentence and B is a sort-m sentence where $m \leqslant n$, then $(A \,\&\, B)$ and $(B \,\&\, A)$ are sort-n sentences.

We use 'PREDBOX' to refer both to the language described above and to the logic based on it. PRED is the language of first order predicate logic whose non-logical vocabulary is comprised of

i) for every natural number i, an individual variable v_i
ii) for all natural numbers i and n, an n-ary predicate letter P_i^n.

PREDBOX ***with identity*** is the language obtained from PREDBOX by eliminating identification in favor of a 0-ary connective I ('identity') of sort 2. Many of our remarks about PREDBOX will apply equally to PREDBOX with identity. Those which apply specifically to the latter system will be enclosed in square brackets. We use $A^n, B^n, C^n, \ldots$ to range over sort-n sentences of PREDBOX — sometimes dropping the superscripts — and $\varphi, \psi, \chi, \ldots$ to range over formulas of PRED.

A PREDBOX model is a structure $(\bigcup D^a, V)$ where $\bigcup D^a$ is the set of all finite or denumerably infinite sequences of members of a nonempty set D, and V assigns to each sort-n sentence letter a subset of the set D^n of length-n sequences of members of D.

Sentences of PREDBOX are evaluated at members of $\bigcup D^a$. The truth value of A^m at a sequence $\boldsymbol{d}$ depends only on the first m terms of $\boldsymbol{d}$. If the length of $\boldsymbol{d}$ is less than m then it makes no sense to say that A^m is true or false at $\boldsymbol{d}$.

NOTATION. If $\boldsymbol{d}$ is a sequence of length at least k then $\boldsymbol{d}_k^+$ is the sequence obtained by deleting the first k terms of $\boldsymbol{d}$.

DEFINITION. Let $M = (\bigcup D^a, V)$ be a PREDBOX model, $\boldsymbol{d} \in \bigcup D^a$, and $m \leqslant$ length $(\boldsymbol{d})$. The notion A^m is true at $\boldsymbol{d}$ in M $((M, \boldsymbol{d}) \vDash A^m)$ is defined by the following clauses: (We will supress the model M when no confusion results.)

i. A is p_i^m.
Then $\boldsymbol{d} \vDash A$ iff $\langle d_1, \ldots, d_m \rangle \in V(p_i^m)$.

ii. A is $-B$.
Then $\boldsymbol{d} \vDash A$ iff not $\boldsymbol{d} \vDash B$.

iii. A is $B \,\&\, C$.
Then $\boldsymbol{d} \vDash A$ iff $\boldsymbol{d} \vDash B$ and $\boldsymbol{d} \vDash C$.

iv. A is $\Box B$.
Then $\boldsymbol{d} \vDash A$ iff $\langle d, d_2, \ldots, d_m \rangle \vDash B$ for all d in D.

v. A is ϱB.
Then $\boldsymbol{d} \vDash A$ if $\langle d_m, d_1, \ldots, d_{m-1} \rangle \vDash B$.

vi. A is σB and $m \geqslant 2$.
Then $\boldsymbol{d} \vDash A$ iff $\langle d_2, d_1, d_3, \ldots, d_m\rangle \vDash B$.

vii. A is σB^1.
Then $\boldsymbol{d} \vDash A$ iff $\boldsymbol{d} \vDash B$.

viii. A is $[||]B$.
Then $\boldsymbol{d} \vDash A$ iff $\langle d_1, d_1, \ldots, d_m\rangle \vDash B$.

[ix. A is I.
Then $\boldsymbol{d} \vDash A$ iff $d_1 = d_2$.]

The remaining classical connectives are defined in the usual way. In addition we define the sort-n constants $\top^n$ (truth) and $\bot^n$ (falsity) as $p_1^n \sim p_1^n$ and $p_1^n \& p_1^n$ respectively. [In PREDBOX with identity $[||]A$ is defined as $-\Box-(I \& A)$.]

Notice that the truth condition for each non-classical operator resembles he truth conditions for modal operators in the Kripke semantics in the sense that a complex sentence is true at a sequence exactly when its constituent sentence is true at all appropriately "related" sequences. We can deduce from the properties of the relation associated with each operator some of the logical characteristics of that operator. $\Box A$, for example, is true at a sequence $\boldsymbol{s}$ if A is true at all sequences which are identical with $\boldsymbol{s}$ at every place but the first. Since "identity of tails" is an equivalence relation, $\Box$ will satisfy the theorems of **S5**. Similarly, the relations associated with ϱ, σ and $[||]$, when their truth conditions are written in the relational form, have the property that each sequence is related to exactly one sequence. Hence if $\bigcirc$ is any of these operators $-\bigcirc A \leftrightarrow \bigcirc - A$ will be a theorem.

DEFINITION. $M = (\bigcup D^a, v)$ is a *model for* A^n (written $M \vDash A^n$) if $(M, \boldsymbol{d}) \vDash A^n$ for all $\boldsymbol{d}$ in $\bigcup D^a$ such that length $(\boldsymbol{d}) \geqslant n$. A is *valid* if every PREDBOX model is a model for A.

Special Abbreviations. In the following we assume $m = \max(n, k)$

1) $\varrho_k A^n =_{df} \varrho^{(m+1)-k}(\varrho\sigma)^{k-1}(A^n \& \top^k)$

2) $\varrho_k^{-1} A^n =_{df} \varrho_k^{k-1} A^n$

3) $$\delta_{jk} A =_{df} \begin{cases} \varrho_k \varrho_j [||] \varrho_j^{-1} \varrho_k^{-1} A & \text{if } j > k \\ \varrho_k \varrho_{j+1} [||] \varrho_{j+1}^{-1} \varrho_k^{-1} A & \text{if } j < k \\ A & \text{if } j = k \end{cases}$$

4) $\delta_{(j_1, \ldots, j_k)} A =_{df} \delta_{(j_1, j_2)} \delta_{(j_2, j_3)} \ldots \delta_{(j_{k-1} j_k)} A$.

PROPERTIES. *Suppose* $\boldsymbol{d}$ *is a sequence of length at least* m.[5]

a) $\varrho_n A^n$ *and* $\varrho_n^{-1} A^n$ *are sort-n*
$\delta_{jk} A^n$ *is sort* $\max(j, k, n)$, *and* $\delta_{(j_1, \ldots, j_k)} A^n$ *is sort* $\max(n, j_1, \ldots, j_k)$

[5] '$\boldsymbol{de}$' denotes the concatenation of $\boldsymbol{d}$ and $\boldsymbol{e}$. When boldface letters are used to denote sequences, the corresponding plainface letters with subscripts denote terms in those sequences.

b) $\boldsymbol{d} \vDash \varrho_k A^n$ *iff* $\langle d_k, d_1, \ldots, d_{k-1}\rangle d_k^+ \vDash A^{n*}$

c) $\boldsymbol{d} \vDash \varrho_k^{-1} A^n$ *iff* $\langle d_2, \ldots, d_k, d_1\rangle d_k^+ \vDash A^n$

d) $\boldsymbol{d} \vDash \delta_{ij} A$ *iff* $\langle d_1, \ldots, d_{j-1}, d_i\rangle d_j^+ \vDash A^n$

e) $(M, \boldsymbol{d}) \vDash \delta_{(j_1,\ldots,j_k)} A$ *iff* $(M, \boldsymbol{d}') \vDash A$ *where* $\boldsymbol{d}'$ *is the result of replacing the* j_i^{th} *term of* $\boldsymbol{d}$ *for* $2 \leqslant i \leqslant k$ *by the* j_1^{st}.

DEFINITION. Let S be a finite sequence of natural numbers. We define the sentence $\gamma s A$ by induction on the length of $\boldsymbol{s}$.

$$\gamma\langle s_1\rangle A =_{df} \varrho_{s_1} A$$
$$\gamma\langle s_1, \ldots, s_n\rangle A =_{df} \varrho_{s_n} \gamma\langle t_1, \ldots, t_{n-1}\rangle A$$

where, for $1 \leqslant i \leqslant n-1$, $t_i = s_i$ if $s_i \geqslant s_n$ and $t_i = s_i+1$ if $s_i < s_n$.

PROPERTY. *Suppose* $M = (\bigcup D^a, V)$ *is a* PREDBOX *model,* $\langle d_1, \ldots, d_n\rangle$ *is a sequence of members of* D, k *is a natural number no larger than* n, *and* $\langle t_1, \ldots, t_k\rangle$ *is a one-one sequence of natural numbers such that, for* $1 \leqslant i \leqslant k$, $n-k < t_i \leqslant n$. *Then* $(M, \langle d_1, \ldots, d_n\rangle) \vDash \gamma\langle t_1, \ldots, t_k\rangle A^n$ *iff* $(M, \langle d_{t_1}, \ldots, d_{t_k}, d_1, \ldots, d_{n-k}\rangle) \vDash A^n$.

DEFINITION. Let $\boldsymbol{s} = \langle s_1, \ldots, s_n\rangle$ be a sequence of natural numbers $\leqslant m$.

We define the one-one sequence $^*\boldsymbol{s} = \langle {}^*s_1, \ldots, {}^*s_n\rangle$ as follows:

$^*\langle s_1\rangle = \langle s_1\rangle$
$^*\langle s_1, \ldots, s_{k+1}\rangle = {}^*\langle s_1, \ldots, s_k\rangle\langle t\rangle$ where $t = s_{k+1}$
if s_{k+1} doesn't occur in $^*\langle s_1, \ldots, s_k\rangle$ and
t is the first term of $\langle 1, \ldots, \max(m, n)\rangle$ which
doesn't occur in $^*\langle s_1, \ldots, s_k\rangle$ or $\boldsymbol{s}$ otherwise.

Notice that no term of $^*s$ is larger than $\max(m, n)$.

For all i, $1 \leqslant i \leqslant n$, let p_i be the number of occurrences of s_i in $\boldsymbol{s}$. For all j, $1 \leqslant j \leqslant p_i$, we write $\boldsymbol{S}(i, j) = k$ to indicate that k is the j^{th} place in $\boldsymbol{s}$ at which s_i occurs. (So, for example, $\boldsymbol{S}(1, 1) = 1$.)

$$\Theta_{\boldsymbol{S}} A =_{\text{df}} \gamma_{*S}\, \delta_{(S(1,1),\ldots,S(1,p_1))} \ldots \delta_{(S(1,1),\ldots,S(n,p_n))} A$$

PROPERTIES. i) *Sort* $(\theta_{\boldsymbol{S}} A^m) = \max(m, n)$

ii) $\langle k_1, \ldots, k_n\rangle \vDash \Theta_{\boldsymbol{S}} A^n$ *iff* $\langle k_{s_1}, \ldots, k_{s_n}\rangle \vDash A^n$.

With each PREDBOX model $M = (\bigcup D^a, V)$ we associate the PRED model $\mathfrak{A}(M)$ which has domain D and which interprets each n-place predicate letter P_i^n by $V(p_i^n)$. Conversely, if $\mathfrak{A} = (U, \overline{P_1^1}, \ldots)$ is a PRED model we associate a PREDBOX model $M(\mathfrak{A})$ with it: namely (U^a, V) where $V(p_i^n) = \overline{P_i^n}$. Notice that $M(\mathfrak{A}(M)) = M$ and $\mathfrak{A}(M(\mathfrak{A})) = \mathfrak{A}$. We shall also need to translate members of $\bigcup D^a$ into assignments So if $\boldsymbol{d} \in \bigcup D^a$, let $A(\boldsymbol{d})$ be an assignment such that $A(v_i) = d_i$ if $i \leqslant$ length $(\boldsymbol{d})$. Conversely if a is an assignment from $\mathfrak{A}$, let $\boldsymbol{d}(a)$ be the sequence whose i^{th} term is $a(v_i)$. Clearly $A(\boldsymbol{d}(a)) = a$ and $\boldsymbol{d}(A(\boldsymbol{d}))$ is either $\boldsymbol{d}$ or an extension of $\boldsymbol{d}$.

We now show that PREDBOX (with identity) has the same expressive power as PRED (with identity).

NOTATION. 1) $\varphi[\![x_1, \ldots, x_n/y_1, \ldots, y_n]\!]$ is the result of simultaneously substituting $y_1, \ldots, y_n$ for all occurrences (free and bound) of $x_1, \ldots, x_n$, respectively, in φ.

2) $\varphi\langle x/y\rangle$ is defined by induction as follows

i) $P_i^n x_1, \ldots, x_n\langle x/y\rangle = P_i^n x_1, \ldots, x_n[\![x/y]\!]$.

ii) $(-\psi)\langle x/y\rangle = -(\psi\langle x/y\rangle)$.

iii) $(\psi \,\&\, \chi)\langle x/y\rangle = \psi\langle x/y\rangle \,\&\, \chi\langle x/y\rangle$

iv) $(\forall v\psi)\langle x/y\rangle = \begin{cases} \forall v(\psi\langle x/y\rangle) \text{ if } v \neq x \text{ and } v \neq y \\ \forall x\psi \text{ if } v = x \\ (\forall y\psi)[\![x, y/y, x]\!] \text{ if } v = y \end{cases}$

DEFINITIONS. The map f: PREDBOX$\to$PRED is defined as follows:

i. $f(p_i^n) = P_i^n v_1 \ldots v_n$

ii. $f(-A) = -f(A)$

iii. $f(A \,\&\, G) = f(A) \,\&\, f(G)$

iv. $f(\varrho A^n) = f(A^n)[\![v_1, \ldots, v_n/v_n, v_1, \ldots, v_{n-1}]\!]$

v. $f(\sigma a^n) = \begin{cases} f(A^n)[\![v_1, v_2/v_2, v_1]\!] \text{ if } n \geqslant 2 \\ f(A^n) \text{ otherwise} \end{cases}$

vi. $f(\Box A^n) = \forall v_1 f(A^n)$

vii. $f([\|]\,A^n) = f(A^n)\langle v_2/v_1\rangle$

[viii. $f(I) = (v_1 = v_2)$].

The map g: PRED$\to$PREDBOX is defined as follows:

i. $g(P_j^n v_{k_1} \ldots v_{k_n}) = \Theta\langle k_1, \ldots, k_n\rangle p_j^n$

ii. $g(-\varphi) = -g(\varphi)$

iii. $g(\varphi \,\&\, \psi) = g(\varphi) \,\&\, g(\psi)$

iv. $g(\forall v_j\varphi) = \varrho_j \Box \varrho_j^{-1} g(\varphi)$

[v. $g(v_j = v_k) = \begin{cases} \varrho_j \varrho_k I \text{ if } j < k \\ \varrho_k \varrho_j I \text{ if } j > k \\ \top^j \quad \text{ if } j = k]. \end{cases}$

THEOREM. 1) *If* $\mathfrak{A} \models^a \varphi$ *then* $(M(\mathfrak{A}), \boldsymbol{d}(a)) \models g(\varphi)$

2) *If* $(M, \boldsymbol{d}) \models B$ *then* $\mathfrak{A}(M) \models^{A(\boldsymbol{d})} f(B)$.

COROLLARY.[6] 1) $\models \varphi \leftrightarrow f(g(\varphi))$ 3) *If* $\models \varphi$ *then* $\models g(\varphi)$

2) $\models A \leftrightarrow g(f(A))$ 4) *If* $\models A$ *then* $\models f(A)$.

[6] For arguments that the existence of translations satisfying these four conditions constitutes equivalence see [9] pp. 67-97. In the remainder of this paper 'equivalence' will always be used in this sense.

III. Axiomatization

This section contains a completeness proof for PREDBOX which resembles both the Makinson-style completeness proofs for propositional modal logics and the Henkin-style proofs for classical predicate logic.[7] The strategy is to construct a PREDBOX model which satisfies an arbitrary consistent sentence A by extending $\{A\}$ to a maximal consistent set of sentences Γ and constructing from Γ a sequence at which all and only the members of Γ are true.

DEFINITION. The *theorems* of PREDBOX (with identity) are the members of the smallest set containing

1) All tautologous sentences
2) All instances of A1-A11 (A1-A15) below and closed under:
3) Tautologous consequence
4) Rules R1, R2 below.

A1. $-\bigcirc A \leftrightarrow \bigcirc - A$ for $\bigcirc = [|], \varrho, \sigma$

A2. $\bigcirc(A \,\&\, B) \leftrightarrow (\bigcirc A \,\&\, \bigcirc B)$ for $\bigcirc = \Box, \varrho, \sigma, [|]$

A3. $\Theta_{\langle k, k_2, \ldots, k_n \rangle} \Box A^n \leftrightarrow \Theta_{\langle j, k_2, \ldots, k_n \rangle} \Box A^n$

A4. $\delta_{ij}\, \delta_{ji} A \leftrightarrow \delta_{ij} A$

A5. $\delta_{ij}\, \delta_{jk} A \leftrightarrow \delta_{ik}\, \delta_{ij} A$

A6. $\delta_{jk_i} \Theta_{\langle k_1, \ldots, k_n \rangle} A \rightarrow \Theta_{\langle k_1, \ldots, k_{i-1}, j, k_{i+1}, \ldots, k_n \rangle} A$
$\Theta_{\langle k_1, \ldots, k_n \rangle} \varrho_j A^m \leftrightarrow \Theta_{\langle j, k_2, \ldots, k_n \rangle} A^m$

A7. $\Theta_{\langle k_1, \ldots, k_n \rangle} [|] A \leftrightarrow \Theta_{\langle k_1, k_1, k_3, \ldots, k_n \rangle} A$

A8. $\Theta_{\langle k_1, \ldots, k_n \rangle} \Box A \rightarrow \Theta_{\langle k_1, \ldots, k_n \rangle} A$

A9. $\Theta_{\langle k_1, \ldots, k_n \rangle} \varrho A^n \leftrightarrow \Theta_{\langle k_n, k_1, \ldots, k_{n-1} \rangle} A^n \quad (n \geqslant 2)$
$\varrho A^1 \leftrightarrow A^1$

A10. $\Theta_{\langle k_1, \ldots, k_n \rangle} \sigma A^n \leftrightarrow \Theta_{\langle k_2, k_1, \ldots, k_n \rangle} A^n \quad (\mathrm{n} \geqslant 2)$
$\sigma A^1 \leftrightarrow A^1$

A11. $\Theta_{\langle 1, \ldots, n \rangle} A^n \leftrightarrow A^n$

[A12. $\Theta_{\langle k, k \rangle} I$]

[A13. $\Theta_{\langle i, j \rangle} I \rightarrow \Theta_{\langle j, i \rangle} I$]

[A14. $(\Theta_{\langle i, j \rangle} I \,\&\, \Theta_{\langle j, k \rangle} I) \rightarrow \Theta_{\langle i, k \rangle} I$]

[A15. $(\Theta_{\langle j, k_i \rangle} I \,\&\, \Theta_{\langle k_1, \ldots, k_n \rangle} A^n) \rightarrow \Theta_{\langle k_1, \ldots, k_{i-1}, j, k_{i+1}, \ldots k_n \rangle} A^n$]

R1. If $\vdash A$ then $\vdash \bigcirc A$ for $\bigcirc = \varrho, \sigma, \Box, [|]$.

R2. If $\vdash A^m \rightarrow \Theta_{\langle k_1, \ldots k_n \rangle} B^n$ then $\vdash A^m \rightarrow \Theta_{\langle k_1, \ldots, k_n \rangle} \Box B^n$
(provided $k_1 > \max(m, n, k_2, \ldots, k_n)$).

The axioms and rules have been chosen to facilitate the completenes proof. It is easy to verify that the theorems of PREDBOX are all valid

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[7] Illustrations of these can be found, for example, in [11] (pp. 1-39) and [4] (pp. 128-136).

DEFINITIONS. A set Γ PREDBOX is *consistent* if it contains no finite subset $\{A_1, \ldots, A_n\}$ such that $\sim(A_1 \,\&\, \ldots \,\&\, A_n)$ is a theorem. Γ is *maximal consistent* if it is consistent and is not strictly contained in another consistent set. Γ is *saturated* if it is maximal consistent and, in addition, whenever $\Theta_{\langle k_1,\ldots,k_n\rangle}\Box A \notin \Gamma$ there is some k such that $\Theta_{\langle k,k_2,\ldots,k_n\rangle} A \notin \Gamma$.

LEMMA. *If $\{A^m\}$ is consistent it can be extended to a saturated set, Γ.*

PROOF: Let $A_1, A_2, \ldots$ be an enumeration of PREDBOX such that, whenever A_i is of the form $\Theta_{\langle k_1,\ldots,k_n\rangle}\Box B^n$ A_{i+1} is $\Theta_{\langle k,\ldots,k_n\rangle}-B^n$ where $k = 1+\sup\{sort(A_j)\colon j \leqslant i\}$. Let $\Gamma = \bigcup_i \Gamma_i$ where $\Gamma_0 = \{A^m\}$ and Γ_{i+1} is $\Gamma_i \cup \{A_{i+1}\}$ if this is consistent and $\Gamma_i \cup \{-A_{i+1}\}$ otherwise. It is easy to check that Γ is maximal consistent. To see that it is saturated suppose that $A_j = \Theta_{\langle k_1,\ldots,k_n\rangle}\Box B^n$ and $A_j \notin \Gamma$. We claim that $A_{j+1} \in \Gamma$. For if not $\Gamma_j \cup \{A_{j+1}\}$ would be inconsistent, so for some $C_1, \ldots, C_m \in \Gamma_j (C_1 \,\&\, \ldots$ $\ldots \,\&\, C_m) \to -\Theta_{\langle k,k_2,\ldots,k_n\rangle} -B^n$ would be a theorem. But by A1, R2, A3 this means that $(C_1 \,\&\, \ldots \,\&\, C_m) \to \Theta_{\langle k_1,\ldots,k_n\rangle} \Box B^n$ is a theorem which violates the maximal consistency of Γ.

THEOREM. *If A^m is consistent there is a model M and a sequence $\mathbf{d}$ such that $(M, \mathbf{d}) \models A^m$.*

PROOF: Let Γ be a saturated set containing A. We define the binary relation $\sim_\Gamma$ on natural numbers as follows:

$$i \sim_\Gamma j \text{ iff } A \leftrightarrow \delta_{ij} A \in \Gamma \text{ for all } A.$$

$$[i \sim_\Gamma j \text{ iff } \Theta_{\langle i,j\rangle} I \in \Gamma.]$$

Axioms 4 and 5 [6, 7 and 8] insure that $\sim_\Gamma$ is an equivalence relation. We use the notation '$[n]$' to denote the equivalence class of n under $\sim_\Gamma$. We define a PREDBOX model $M = \langle \bigcup D^a, V\rangle$ by letting $D = \{[i]\colon i < w\}$ and taking $V(p^n)$ to be $\{\langle [k_1], \ldots, [k_n]\rangle\colon \Theta_{\langle k_1,\ldots,k_n\rangle} p^n \in \Gamma\}$ for each sort-n sentence letter p^n. V is well defined by axiom 10 (axiom 9).

To prove the theorem it suffices, by All, to show that for any sort-n sentence A, $(M, \langle [k_1], \ldots, [k_n]\rangle) \models A$ iff $\Theta_{\langle k_1,\ldots,k_n\rangle} A \in \Gamma$. We do so by induction on the construction of A.

i) If A is a sentence letter the result follows immediately from the Definition of V.

ii) Suppose A is of the form $\Box B$. If $(M, \langle [k_1], \ldots, [k_n]\rangle) \models A$ then, for all k, $(M, \langle [k], [k_2], \ldots, [k\ ]\rangle) \models B$. So, by induction hypothesis $\Theta_{\langle k,k_2,\ldots,k_n\rangle} B \in \Gamma$ for all k. But Γ is saturated, which means $\Theta_{\langle k,k_2,\ldots,k_n\rangle} B \in \Gamma$. Conversely, if $(M, \langle [k_1], \ldots, [k_n]\rangle \models A$, then for some k, $(M, \langle [k], [k_2], \ldots$ $\ldots, [k_n]\rangle) \models B$ and, by induction hypothesis, $\Theta_{\langle k,k_2,\ldots,k_n\rangle} B \notin \Gamma$. By A8 $\Theta_{\langle k,k_2,\ldots,k_n\rangle} A \notin \Gamma$ and, by A3 $\Theta_{\langle k_1,\ldots,k_n\rangle} A \notin \Gamma$.

The remaining cases are straightforward and are left to the reader. This completes the proof of the theorem.

Strong completeness.

A set Γ of sentences is said to be satisfied in a model $M = \langle \bigcup D^a, V\rangle$ if there is some sequence $\boldsymbol{d}$ in $\bigcup D^a$ at which all the members of Γ are true. (Notice that if there is no bound on the sorts of members of Γ, then Γ will be satisfied only at infinite sequences.) As in the case of PRED, a proof of *strong* completeness (i.e., that every consistent set is satisfied by some model) is complicated by the fact that some consistent sets *cannot* be extended to saturated ones. The most common strategy to circumvent this problem in the case of PRED is to augment the language with a new stock of variables. This procedure has no analog in PREDBOX, however, so we must turn to another trick.[8] Given a consistent set Γ of sentences, let $2\Gamma = \{\Theta\langle 2, 4, \ldots, 2n\rangle A : A \in \Gamma\}$ 2Γ can be extended to a saturated set by a construction like the one given above, and the model for 2Γ can easily be converted into a model for Γ.

IV. Fragments

A. *n*-predbox

For every natural number n an *n-model* is a structure $\langle D^n, V\rangle$ where D^n is the set of all n-tuples of members of the non-empty set D and V is a function which assigns a subset of D^n to every sort-n sentence letter of PREDBOX. We define truth and validity of sort-n sentences in n-models in the obvious way. For each n, the n-models and sort-n sentences of PREDBOX constitute a special modal system which we call n-PREDBOX. (A^n is valid in the class of n-models if and only if it is valid in the class of PREDBOX models, so we can think of n-PREDBOX as the sort-n fragment of PREDBOX.) 1-PREDBOX is just ***S5***. (That is $\Box$ obeys the ***S5*** theorems and $\bigcirc A \leftrightarrow A$ is a theorem when $\bigcirc$ is ϱ, σ, or $[|]$.) 2-PREDBOX turns out to be the "basic two-dimensional modal logic," B, of Krister Segerberg.[9] The translations of Section II establish that n-PREDBOX is equivalent to the n-variable fragment of n-adic predicate logic, i.e., to the class of valid formulas containing only n-place predicate letters and variables from among $v_1, \ldots, v_n$.

Rule R2 in our axiomatization of PREDBOX allows sort-n sentences to be derived from sentences of sorts larger than n. Hence we cannot assume that n-PREDBOX is axiomatized by restricting the axioms and rules for PREDBOX to the language of n-PREDBOX. In our completeness proof for PREDBOX we constructed an *infinite* sequence at which A was true. Whenever a formula of the form B was false at the sequence we could construct a sequence at which B was false by bringing sufficiently

[8] See [13] pp. 142-149 for an illustration of the use of this kind of strategy in proving completeness of PRED.

[9] See [19]. Segerberg's system and 2-PREDBOX are discussed in [10].

distant terms to the head of the sequence. This strategy is not applicable in the case of n-PREDBOX where all sentences are evaluated at sequences of fixed finite length. In a planned sequel to this paper it will be shown that our completeness proof can be adapted to n-PREDBOX without identity. The completeness problem for n-PREDBOX with identity remains open.[10]

The decidability questions for n-PREDBOX, on the other hand, are all answered. It follows from [19] that 2-PREDBOX is decidable[11] and from [8] that n-variable n-adic logic (and hence n-PREDBOX) is undecidable for $n > 2$.

B. Unidentified predbox

The quantifiers and variables of PRED are replaced in PREDBOX by four distinct operators. By separating the roles played by each of these operators we might hope to gain a new perspective on predicate logic. In this subsection we establish the undecidability of the [|]-free-fragment of PREDBOX, hereafter referred to as *unidentified* PREDBOX. From our translations between PRED and PREDBOX it can be seen that unidentified PREDBOX is equivalent to the class of valid formulas of predicate logic which contain no atomic subformula with more than one occurrence of the same variable. We will show that the decision problem for PREDBOX can be reduced to that for unidentified PREDBOX.

If A is a sentence of PREDBOX let p_A be a sort-2 sentence letter of PREDBOX which does not occur in A. Let $E(A)$ be the set containing i, ii, and iii below and all instances of iv such that B^n is of the form $\varrho_k q^n$ where q^n is a sentence letter occurring in A and $k \leqslant n$.

i. $p_A \to \sigma p_A$
ii. $(\varrho_3 p_A \,\&\, p_A) \to \sigma \varrho_3 p_A$
iii. $\Box \sigma - \Box - p_A$
iv. $\varrho_{n+1} p_A \to (B^n \leftrightarrow \varrho_{n+1} \sigma \varrho^n_{n+1} B^n)$.

Notice that $E(A)$ is always finite and no member of $E(A)$ contains any occurrences of identification. We use the notation '$KE(A)$' to denote the conjunction of the members of $E(A)$.

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[10] The problem is somewhat similar to one discussed by Leon Henkin in [6]. In the languages Henkin considers formulas may contain only the variables $v_1, \ldots, v_n$, but they may contain predicate letters of any degree. Henkin points out that in the usual axiomatizations of predicate logic, proofs of n-variable formulas may require formulas with more than n-variables. For example there seems to be no proof involving only x and y of the formula

$$(\exists x)(x = y) \,\&\, \exists x\, Gxy) \to \forall x (x = y \to \exists x\, Gxy).$$

He asks, among other questions, whether n-variable predicate logic can be axiomatized by a finite set of n-variable schemata.

[11] See also [20] and [1] pp. 88-89.

LEMMA. *If M is a* PREDBOX *model such that $M \models KE(A)$ then*

i) $V(p_A)$ *is an equivalence relation*

ii) *If $\langle d_1, e\rangle \in V(p_A)$ and q^n is a sentence letter in A then $(M, \langle d_1, \ldots, d_n\rangle) \models q^n$ iff $(M, \langle d_1, \ldots, d_{i-1}, e, d_{i+1}, \ldots, d_n\rangle) \models q^n$.*

DEFINITION. For all A in PREDBOX we define two maps s_A and t_A from PREDBOX to unidentified PREDBOX as follows:

$$s_A q^n = q^n$$
$$s_A(B \,\&\, C) = s_A B \,\&\, s_A C$$
$$s_A \bigcirc B = \bigcirc s_A B \text{ for } \bigcirc = \Box, \sigma, \varrho, -$$
$$s_A[|]B = p_A \,\&\, s_A B$$
$$t_A B = KE(A) \,\&\, s_A B.$$

DEFINITIONS. Suppose $M = (\bigcup D^a, V)$ is a PREDBOX model.

i) Let M_A be the model $(\bigcup D^a, V')$ where $V'(p_n)$ is the identity relation on D and $V'(p^n) = V(p^n)$ for $p^n \neq p_A$.

ii) If $\sim$ is an equivalence relation on D, let $M/\sim$ be the structure $(\bigcup C^a, U)$ where $C = D/\sim$ and $U(p^n) = \{\langle [d_1], \ldots, [d_n]\rangle \mid \langle d_1, \ldots, d_n\rangle \in V(p_n)\}$. ($[d]$ is the equivalence class of d under $\sim$.)

LEMMA.

i) *If $M \models A$ then $M_A \models t_A$*

ii) *If $M = (\bigcup D^a, V)$ is a model for t_A then $M/V(p_A) \models A$.*

The theorem below follows immediately.

THEOREM. *Unidentified* PREDBOX *is undecidable.*

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Georgetown University
Washington

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