LOGICAL EXPRESSIONS, CONSTANTS AND OPERATOR LOGIC

For the last 60 years or so the study of logic has been the study of classical predicate logic. Recently, however, a number of authors have suggested that certain philosophical arguments can best be represented in formal systems other than predicate calculus. They have proposed, for example, logics of tense, necessity and possibility, obligation and permission, idealized knowledge and counterfactual implication. The emergence of these "modal" systems has given a new urgency to questions about the nature and foundations of logic. It is not my intention to try to answer any important foundational questions in this paper. I shall, however, try to point out a few simple facts which I think have been overlooked or insufficiently emphasized in the standard accounts of the nature of logic. One consequence of the neglect of these facts, I believe, is that the logical expressions of a language have been identified with its constants. I shall argue that this identification is mistaken. A second consequence, I believe, is that the modal systems have been treated in the wrong way. I shall suggest a way in which modal systems can be treated as theories rather than logics, without changing their syntax.

The paper is organized as follows. Section One comprises an account of the nature of logic like the sort found in the introduction to an elementary logic text. Section Two outlines a framework within which systems of logic can be constructed. Section Three is a discussion of the neglected facts and Section Four, an outline of the alternative treatment for modal logics.

I. The traditional account of the nature of logic

An argument is a set of sentences one of which (the conclusion) is supposed to follow from the others (the premises). Logic is the study of correct arguments, i.e., arguments for which the conclusion must be true if all the premises are. What makes a systematic study of such arguments possible is that a small class of expressions figure prominently in a great variety of arguments. For example, a large class of correct arguments can be obtained by inserting expressions of the appropriate kind into these familiar patterns:

All ______ are ....
is a ______
Therefore is a ....

If ______ then ....
It is not the case that ....
Therefore it is not the case that ______.

Logicians do not, in fact, study directly arguments formulated in English. Instead they consider arguments in an artificial, idealized language which takes from English only the features likely to be important for their purposes. (In this respect logicians are like physicists, say, who study "point-masses" rather than physical objects.) Certain expressions in this artificial language correspond roughly to the expressions like 'all', 'if', and 'it is not the case that' which feature so prominently in good arguments. These are called logical expressions. A sentence (in the artificial language) is said to be a logical consequence of some other sentences if the argument with the latter as premises and the former as conclusion remains correct after any sensible substi -

tution for the expressions which aren't logical. Alternatively a sentence is a logical consequence of some other sentences if the argument with the latter as premises and the former as conclusion remains correct after any sensible change in the denotation of the nonlogical expressions. On either approach the logical expressions are constants. They are the expressions which are held fixed or whose meanings are held fixed while the others are substituted for or reinterpreted. There are exactly seven logical constants, corresponding roughly to the English expressions 'and', 'or', 'not', 'if', 'if and only if', 'all' and 'some'. The reasons that logical consequence and logical truth can be characterized in a convenient and useful sense by these seven constants is not completely understood--there is undoubtedly an element of arbitrariness or convention in the choice. But we should not let this state of affairs hold back our logical investigations. Whatever the reasons for their being singled out, the seven logical constants have provided the basis for a highly successful science of reasoning. The success is particularly striking in an area in which diffucult chains of arguments have always played an important role--mathematics. Typically, mathematical reasoning can be represented in a language containing the seven logical constants which is capable of expressing a couple of relations among one kind of object (for example, the relation which holds between a number at its successor and that which holds among three numbers, the first of which is the sum of the second and third). A set of sentences in such a language which is closed under logical consequence is called a theory. The idea is that sentences in a theory express the facts about a particular subject. Some sentences in a theory, of course, will be logical truths, but some will be true simply because that's the way things are. It is a remarkable fact that many parts of mathematics can be reconstructed as theories based on the standard logic.

II. Framework

For the sake of definiteness I will assume that any system of formal logic can be characterized by a categorially generated language and a class of compositional interpretations for that language. By this I mean that:

i) The expressions of the language are divided into categories.

ii) Complex expressions are generated from simpler ones by rules of the form: "If $e_1,\ldots\ldots,e_n$ are expressions of category $c_1,\ldots\ldots\ldots,c_n$ then $e(e_1,\ldots,e_n)$ is an expression of category $c(c_1,\ldots\ldots,c_n)$" where e and c are functions from expressions to expressions and categories to categories.

iii) Each category is associated with a semantical type which indicates the kind of objects which may be assigned to expressions of that category by an interpretation.

iv) An interpretation assigns objects of the appropriate type to the primitive expressions directly. The object it assigns to a complex expression $e(e_1,\ldots\ldots,e_n)$ is a function of the objects assigned to $e_1,\ldots\ldots,e_n$ by it and other interpretations. Hence with each function for building complex expressions we can associate a rule of interpretation.

In addition I will assume that there is one class of expressions, the sentences, which the interpretations assign truth values, T and F. A sentence is valid if it is assigned T by every interpretation. A primitive expression is a constant with respect to a class of interpretations if it is assigned the same object by every interpretation in the class.

For example, a natural way to formulate the classical predicate calculus within this framework is to list primitive expressions of the following categories: individual variable, quantifier, n-ary predicate (for each natural number n), two-place connective and one-place connective. The appropriate class of interpretations is one which allows the n-ary predicate P to be assigned to any object consistent with its semantical type (viz., any set of n-tuples of individuals) but which requires & to be interpeted by the unique function which takes the value T for argument (T,T) and the value P for all other arguments. Hence & is a constant and P is not.

III. Facts

An obvious fact that is worth emphasizing is the following:

1) The creator of a logical system can choose which expressions are to be held constant.

The choice depends both on the purposes for which he devises the system and his attitudes towards the system's subject matter.

A formalist interested in set theory, for example, is likely to consider a theory of sets in which $\in$ can receive any interpretation consistent with a set of axioms. A realist about sets, on the other hand, might consider only models whose domain comprises subsets of the cumulative hierarchy and which interpret $\in$ as the membership relation between sets.

Someone interested in characterizing logical truth will presumably restrict the interpretations of the logical expressions. But in this case the question of which expressions are logical must already have been decided. For example, someone who believes '=' is logical will prefer the predicate calculus with identity in which that symbol is always interpreted by the identity relation. Someone who believe '=' is nonlogical will prefer a first order theory of equality in which it may receive any interpretation satisfying the equality axioms, and in particular, by any relation holding between pairs of objects not distinguishable by properties expressible in the language.

Philosophers interested in the semantics of natural language have dealt with systems in which the interpretations of 'the' and 'tomorrow' are restricted. If they want their systems to explain the (nonlogical) inference from 'John is a bachelor' to 'John is unmarried' then presumably the interpretations of 'bachelor' and 'married' will be fixed. Again, the creator of the system <u>chooses</u> his constants to fit his needs and convictions.

Someone interested in establishing the independence of logical postulates or connectives might choose to allow even the interpretation of the Boolean connectives to vary.

A fact which may be a little less obvious than 1) is the following:

> 2) Not all logical expressions of calssical predicate logic are constants.

In classical logic an interpretation is specified by a <u>model</u> $(D, \overline{P}_1, \overline{P}_2, \ldots, \overline{c}_1, \overline{c}_2, \ldots)$ and an <u>assignment</u> $(d_1, d_2, \ldots)$. $\overline{P}_i$ and $\overline{c}_j$ are the objects denoted by the i'th predicate and the j'th individual constant respectively under the interpretation. The logical connectives &, v, -, $\rightarrow$, $\leftrightarrow$ are genuine constants. Since their interpretation cannot vary there is no need to specify it in the model. But the quantifiers' interpretations are specified--by the very first coordinate of the model. The interpretation of $\forall$ can be regarded as the function f from individuals to truth values such that f(X) = T if and only if X contains D. Similarly the interpretation of $\exists$ is the function f from individuals to truth values such that D f(X) = T if and only if X overlaps D. In each case the function which interprets the quantifier varies from model to model. Hence $\forall$ and $\exists$ are not really constants. This reflects the fact that the truth of a sentence like 'Everything is matter' depends not only on what we count as matter, but on what we count as "things."

The quantifiers of second order logic are even less like constants. To interpret them we must specify not only a domain D of individuals but in addition a set of subsets of D over which the predicate quantifiers may range.

The last observation suggests another point, namely:

3) Constancy is a matter of degree.

Although the interpretations of $\forall$ and $\exists$ can vary, they cannot be just anything. Quantifiers are expressions of the type interpreted by functions from sets of individuals to truth values, but not any such function will do. If $X \subset Y$, f(Y) = F and f(X) = T, for example, then f cannot interpret $\forall$. If the class of interpretations allows an expression to be assigned any object consistent with its type we call that expression schematic. In classical logic, predicates and individual constants are purely schematic, truth functional connectives are pure constants and quantifiers lie somewhere in between. Second order quantifiers are more schematic than first order quantifiers, but more constant than predicates.

1) - 3) above suggest that constancy is not a good criterion of logicality. The most that can be said about the connection between the two notions seems to be the following: If we are interested in characterizing the logical truths we will consider systems in which the logical expressions are not treated schematically.

A final point that deserves emphasis is the following:

> 4) A logical expression need not be of any particular grammatical category.

For example the predicate--call it existence--which is true of all individuals and false of none is surely logical. It is after all "definable" in elementary logic from a 0-ary connective that is routinely listed with the logical constants: $\forall x(Ex \leftrightarrow T)$. On the other hand we would not want to classify as logical an expression representing 'At the next full moon it will be the case that...' simply because it happens to be the kind of expression which attaches to sentences to form sentences.

The idea that logical expressions should be sentential operators gains some plausibility from the argument that since these expressions can be applied to all sentences, they must be appropriately general. Tarski's Introduction to Logic, for example, contains the following passage:

> All of these words ('not,' 'and,' 'or,' 'if...then') are well known to us from everyday language and serve to build up compound sentences from simpler ones. In grammar they are counted among the so-called sentential conjunctions. If only for this reason, the presence of these terms does not represent a specific property of any particular science.[1]

But, in fact, there are other general expressions, just as there are very specific operators.

A more detailed account of the connection between grammar and logic has been given by W.V. Quine.[2] Quine distinguishes between two types of expressions--particles and lexicon. Particles are expressions which are not classed initially in any of the categories, but which can enter a complex expression in the course of its construction. On the formulation given previously predicate logic would have no particles at all, but it is easy to imagine another formulation in which &, v, $\rightarrow$ and $\leftrightarrow$ are all particles, there being no category of binary sentential operator, but instead four different rules by which a complex sentence can be built from two simpler ones. This example shows that the particles and lexicon of a language are not uniquely determined by the expressions of that language. Quine suggests that a class of expressions which are interchangeable "salve congruitate" should be considered a category of the lexicon if it is infinite or if we don't care to be definite about which members of the class are in the language. A logical truth is one which is true under all category preserving substitutions for lexical atoms. The logical expressions could presumably be characterized as the particles which occur essentially in logical truths.[3] Quine's defense of this characterization includes a passage similar to the one from Tarski quoted above. "The lexicon is what caters distinctively to special tastes and interests. Grammar and logic are the central facility, serving all comers."[4] Notice, however, that this view does <u>not</u> require that particles (and hence that logical expressions) be of any particular grammatical category. We can create languages in which predicates, predicate modifiers or even sentences are treated as particles. And, as we shall see, we can equally consider languages in which sentential conjunctions are treated as lexical items.

It is important to keep in mind that this discussion concerns the grammar of the artificial language, not that of English. I don't think it is crucial that these grammars be alike. Gilbert Harman[5] has discussed the task of providing a "logical form" for English sentences. Among the principles that Harman says a theory of logical forms should satisfy is that it be "compatible with syntax." For example, purely syntactic evidence leads him to the conclusion that the English word 'if' is not a conjunction like 'and' and 'or,' but a complementizer like 'that' and 'whether.' The logical form of 'if...then ___' sentences is not $A \rightarrow B$, but something like Iab where I is a predicate (perhaps like implication) and a and b are names (perhaps denoting propositions). Predicates are members of large, open-ended classes and hence nonlogical. The principle of modus ponens, depending as it does on the interpretation of a predicate, is not a logical principle. If Harman's task of providing a logical form for all English sentences were the same as the task of constructing a useful logic (i.e., a useful codification of principles of reasoning) this would seem to be a reductio ad absurdem of his position.[6]

IV. Operator Logic

A failure to appreciate the four facts listed above, it seems to me, confuses discussions of the status of modal systems. There is a tendency, for example, to argue that modalities can't be logical expressions because they aren't constants. This reasoning, as we have seen, would banish quantifiers from logic as well as modalities.[7] There is also a tendency to insist that modal logics must be treated in one of two equally unpalatable ways. Either modalities are treated as sentential operators and modal systems as "logics" in their own right, or modalities are treated as predicates of sentences or propositions and modal systems as theories of classical predicate logic. The first view clashes with the intuition that logic is universal, i.e., that it codifies reasoning about all subjects. To speak of a "deontic logic" or an "epistemic logic" is misleading. It suggests that there are special forms of reasoning appropriate to some disciplines, but not others. In fact, however, we don't use, say, "ethical reasoning" to determine our moral obligations, but rather we apply our ordinary everyday reasoning to ethical matters. The second view, on the other hand, requires a recasting of the simple and familiar modal systems into something much less appealing.[8]

I'm not sure whether any modal logics really deserve to be called "logics."[9] Deontic and epistemic logics seem to me to be paradigm examples of theories (theories about philosophically interesting concepts rather than mathematically interesting ones). These theories are not like the familiar theories of classical first order logic, however. Their nonlogical symbols are operators rather than predicates. It is natural to ask, then, on what logic they are based, i.e., what the general principles of reasoning are by which we can get from the (nonlogical) modal axioms to the (nonlogical) modal theorems. I shall try to show that there is a simple answer to this question--they are theories based on something called operator logic. The question of whether an expression is logical or not is a philosophical one and not one whose answer should be forced on us by a formal apparatus. Operator logic will not tell us how to separate the logical from the nonlogical. Whether or not we decide that modal systems are logics, however, operator logic will provide a framework within which they can be preserved intact.

The language of operator logic contains a countable collection of sentence letters, a finite collection of logical operators (presumably the usual Boolean ones[10] and possibly others) and, in addition, for each n, a countable collection of schematic n-ary operators $\boxed{n}_1, \boxed{n}_2, \ldots$ Just as certain predicates (like $\in$) play special roles in first order theories, so certain operators (like $\Box$) may play special roles in theories of operator logic. This special role will be indicated by specifying sets of axioms rather than by restoring the interpretations of the operators directly. It should be emphasized that neither the sentence letters nor the operators are variables. The expression $\Box p \rightarrow p$ of operator logic is like the expression --Fa $\rightarrow$ Fa of first order logic in that neither contains anything that can be bound by a quantifier.

The semantics for operator logic is designed to insure that logical truths are those whose truth hinges on the interpretations of the logical expressions. A model is a four-tuple (P,t,O,V) where P is a nonempty set (the propositions), t is a function from P to the set T,F of truth values, O is a function which assigns to each n-ary operator $\boxed{n}$ a function $\overline{\boxed{n}}$ from n-tuples of propositions to propositions, and V is a function from the set of sentence letters to P. Every sentence expresses a proposition and every proposition has a truth value. The proposition expressed by A in the model M=(P,t,0,V) (written: $|A|_M$) is defined inductively:

i) $|p|_M = V(p)$

ii) $|\Box A_1 \ldots A_n|_M = \overline{\overline{\Box}}(|A_1|_M, \ldots, |A_n|_M)$ (where $\Box$ is an n-ary operator).

A proposition X is true in M if t(X) = T. A sentence A is true in M (written: $M \models A$) if the proposition expressed by A in M is true in M. Otherwise A is false in M.

For each logical operator we add certain restrictions on 0 and t. For example, if & is among the logical operators we might require that $t(\overline{\overline{\&}}(X,Y)=T$ if and only if $t(X)=t(Y)=T$. A model which satisfies the appropriate restrictions on the interpretations of $-$, $\rightarrow$, $\vee$, $\leftrightarrow$, and & is called a classical model.

It turns out that this semantics is equivalent to a more familiar looking possible worlds semantics proposed by M.J. Cresswell.[11] A possible worlds model for operator logic is a five-tuple (W,C,o,0,V) such that W is a nonempty set (the possible worlds), C is a subset of W (the classical worlds), o is a member of N (the actual world), 0 assigns to each n-ary operator $\boxed{n}$ a function $\overline{\overline{\boxed{n}}}$ from n-tuples of subsets of W to subsets of W, and V assigns a subset of W to each sentence letter. If M = (W,C,o,0,V) is a possible worlds model and $w \in W$, we define A is true in M at w ($(M,w) \models A$) as follows.

i) $(M,w) \models p$ if w is a member of V(p).

ii) $(M,w) \models \Box A_1 \ldots A_n$ if w is a member of $\overline{\overline{\Box}}(\|A_1\|, \ldots, \|A_n\|)$ where $\|A\| = \{w \in W: (M,w) \models A\}$.

A is true in M ($M \models A$) if $(M, \circ) \models A$. Again, restrictions are placed on the interpretations of the logical operators. For example, we might require that $\bar{\bar{\&}}(X,Y) \cap C = X \cap Y \cap C$. A model which satisfies appropriate restrictions on all the Boolean connectives is classical. The idea behind Cresswell's semantics is that in certain worlds (the nonclassical ones) even the laws of logic may fail. If $\Box$ is nonlogical the truth values of $\Box A_1 \ldots A_n$ in the actual world might depend on the truth value of $A_1, \ldots, A_n$ in the nonclassical worlds, so we can expect no laws which hinge on nonlogical operators.

Let us call the first kind of model a propositional model. It is not difficult to establish that the two semantics are equivalent in the sense that for every propositional model there is a possible worlds model which makes the same sentences true and for every possible worlds model there is a similarly equivalent propositional model. This can be done by identifying each proposition with a set of possible worlds (the set where the proposition would be true) and each possible world with a set of propositions (the propositions true in that world). The same argument establishes that the classical propositional models are equivalent to the classical possible worlds models.

Sentences like $\Box(p \vee q) \leftrightarrow \Box(q \vee p)$ which are valid in the weakest modal systems are not valid in operator logic. In fact the only sentences of operator logic true in all classical models are the tautologous sentences (i.e., the substitution instances of tautologies). To see this, let A be a non-tautologous sentence. Then there is an assignment α of truth values to the "components" of A which makes A false. We can construct a propositional model to falsify A as follows: Let P be the set of sentences of the language of operator logic. Let O assign to each n-ary operator $\Box$ the function $\overline{\Box}$ such that $\overline{\Box}(A_1,\ldots,A_n) = \Box A_1\ldots A_n$ and, for all sentence letters p, let V(p)=p. Then every sentence expresses itself and, if t is any "classical" assignment of truth values to sentences which agrees with α on the components of A, (P,t,O,V) will be a propositional model which falsifies A.

It should not be surprising that operator logic adds no essentially new theorems to propositional logic. If it did we would suspect that the operators are not really being treated schematically. On the other hand one can imagine adding various nonlogical axioms and rules to form theories of operator logic. One such would be the theory E of the congruence operator axiomatized by the single rule $A \leftrightarrow B \;/\; \Box A \leftrightarrow \Box B$ and known to be complete for the class of possible worlds models for which C = W. Another would be the theory of K of the Kripke operator axiomatized by adding $\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$ to the rule for E.

A completeness result for a modal theory system can be viewed as a kind of representation theorem. It shows that, for many purposes, we don't need to consider all the operator logic models satisfying a modal theory because for any such model there is an elementarily equivalent one of a particularly nice sort. This is analagous to the result that every group is isomorphic to a group of permutations or that every Boolean algebra is isomorphic to an algebra of sets, or that every model for predicate logic is elementarily equivalent to a model whose domain is numbers and whose relations are arithmetical.

It might be possible to interpret operator logic as a theory of first order logic. Predicates would play the role of operators, individuals would be propositions and a special quotation function taking sentences to individual symbols would enable us to get the effect of iterating operators. But this possibility does not mean that operator logic is any less fundamental than first order logic, for the opposite is also true. First order logic can be interpreted as a theory of propositional operator logic--as an extension, in fact, of the theory of four Kripke operators.

Does it make any practical difference whether modal systems are viewed as theories of operator logic or as logics in their own right? Perhaps not, but I suspect that the view that modal systems are intended to supplement or supplant classical logic has encouraged unnecessary antagonism towards them. Certainly the great variety of alethic and deontic modal systems would be embarrassing to someone who thinks of these systems as constituting the logic of necessity and obligation. This embarrassment is removed when the modal systems are viewed as <u>theories</u> characterizing different kinds of necessity and obligation. There are, after all, a number of theories of '$<$' corresponding to different kinds of ordering relations. I also suspect that the view that modal systems are logics influenced the kind of problems which have interested workers in the field. A great deal of the technical work in modal logic is based on an analogy with predicate logic that links box with the logical expressions rather than with the predicates. To mention just one example, there have been extensive investigations of the interpolation property for modal systems.[13] But all this work concerns the existence of interpolants sharing only predicates with the antecedent and consequent of a conditional, rather than interpolants sharing predicates and operators. Finally, since not all of its theories are extension of E, operator logic might provide a framework to deal with concepts that have not been satisfactorily treated as modal systems. It would admit, for example, theories of realistic belief and knowledge, i.e., belief and knowledge not closed under logical equivalence.

The theme of this paper has been that the question of what is logical is a philosophical question which ought not to be decided on purely formal grounds. I have argued in particular that modalities need not be treated either as logical operators or as nonlogical predicates. I have described a framework which I call operator logic within which modal systems can be preserved intact as theories. Operator logic need not contain any new logical principles, but it can accommodate such principles if we decide they are needed. Finally, I have suggested that to view modal systems as theories of operator logic might be more sound philosophically and more fruitful mathematically than to view them in the traditional ways.

FOOTNOTES

[1]A. Tarski, Introduction to Logic (New York: Oxford University Press, 1965) p. 19.

[2]W.V. Quine, Philosophy of Logic (Englewood Cliffs, New Jersey: Prentice Hall, 1970) p. 1.

[3]The qualifier is intended to rule out occurrences like that of 'ouch' in the sentence "Grass is green or grass is not green or Igor said ouch'. See Hinman, Kim and Stich, "Logical Truth Revisted" The Journal of Philosophy, 70 (1968): 495-499. There appears to be some problems with this view of logicality. (A because B) $\rightarrow$ A is true under every appropriate substitution for non-particles so apparently 'because' must be considered logical. Yet traditionally causal sentences have been considered part of science. On the other hand $\forall y\ \exists x\ Fy \rightarrow \forall x\ Fx$ does not remain true when y is replaced by x so it would seem to be nonlogical. The second problem leads Quine to suggest that either the definition of logical truth or the guidelines for separating particles and lexicon be slightly modified.

[4]Quine, op. cit. p. 102.

[5]Gilbert Harman, "Logical Form" Foundations of Language, 9 (1972): 38-65 and "Modus Ponens" dittoed, Princeton University.

[6]In fact I don't think Harman and I are addressing the same issue. Both Aristotle's and Montague's treatments of "universal" sentences would seem to be more compatible with English syntax than the treatment provided by predicate logic. The superiority of predicate logic as a system of logic must stem partly from the fact that it contains simple, unambiguous and

informative sentences like $\forall x \exists y\ Fxy$ and $\exists y \forall x\ Fxy$ which cannot be easily expressed by English sentences. Frege's system was successful partly because he did allow its grammar to depart from that of natural language.

[7]Richmond Thomason, "Philosophy and Formal Semantics" in Leblanc (ed), Truth Syntax and Modality (Amsterdam: North Holland, 1973) criticizes Quine on essentially this point. Dana Scott ("Advice on Modal Logic" in Lambert (ed), Philosophical Problems in Logic (Dordrecht: Reidel, 1970) argues that the S5 necessity operator is logical, but all other intensional operators are not, apparently on the grounds that only the S5 is as constant as the quantifier. Oddly enough, Richard Montague in "Logical Necessity, Physical Necessity, Ethics and Quantifiers," Inquiry, 4 (1960): 259-269, reprinted in Thomason (ed), Formal Philosophy (New Haven: Yale, 1974) took the contrary position that, while all the theorems of K are logical truths, the theorem $\Box A \rightarrow A$ of S5 is nonlogical.

[8]See Harman, "Logical Form" op. cit. pp. 48-51; R. Montague, "Synthetical Treatments of Modality, with Corrolaries on Reflexion Principles and Finite Axiomitizability," Acta Philosophica Fennica, 16 (1963): 153-167, reproduced in Thomason (ed), op. cit.; and G. Niemi, "On the Existence of a Modal Antinomy," Synthese, 23 (1972): 463-476.

[9]For reasons which will not be discussed here I think that the modal systems most likely to be so deserving are the tense systems.

[10]I don't want to be dogmatic on this point, in case it turns out that there are convincing reasons to accept a nonclassical logic and consider, say, classical negation as a theory of operator logic.

[11]M.J. Cresswell, "Intensional Logics and Logical Truths," Journal of Philosophical Logic, 1 (1972): 2-15.

[12]Steven Kuhn, "Quantifiers as Modal Operators," forthcoming in Studia Logica.

[13]See, for example, Kit Fine, "Failures of the Interpolation Lemma in Quantified Modal Logic," Journal of Symbolic Logic, 44 (1979): 201-206; J. Czermak, "Interpolation Theorem for Modal Logics," Journal of Symbolic Logic, 39 (1974) 416; and D. Gabbay, "Craig's Interpolation Lemma for Modal Logics" in Conference in Mathematical Logic, London, 1970 (New York: Springer, 1972), pp. 111-127.