NOTES ON A PROBLEM OF W. V. QUINE

A Working Paper

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I. Introduction

The problem with which this paper is concerned is that of axiomatizing predicate functor logic. Predicate functor logic is a natural, variable-free equivalent of predicate logic which has been developed by Professor Quine in a series of papers including "Towards a Calculus of Concepts" (JSL, 1936), "Variables Explained Away" (<u>Proceedings of the American Philosophical Society</u>, 1960), "Algebraic Logic and Predicate Functors" (<u>Logic and Art</u>, 1971) and "Predicate Functors Revisited" (forthcoming). In this system, the work of the variable is done by an assortment of <u>functors</u>, which attach to predicates to form new predicates. For example pad ("[") and crop ("]") are functors which attach to n-ary predicates to form (n+1)-ary and (n-1)-ary predicates, respectively. If A is an n-ary predicate then $[A$ holds of objects $d_1, \ldots, d_{n+1}$ just in case A holds of $d_1, \ldots, d_{n+1}$ and $]A$ holds of $d_1, \ldots, d_{n-1}$ just in case A holds of $d_o, \ldots, d_{n-1}$ for some object d_o. Functors also do the work of connectives. Thus $A \rightarrow B$ is a predicate which is true of $d_1, \ldots, d_n$ just in case either A is false of $d_1, \ldots, d_n$ or B is true of $d_1, \ldots, d_n$. A predicate is <u>valid</u> if it holds of any sequence of objects. The axiomatization problem is that of axiomatizing the valid predicates. Quine considers the problem in the "Algebraic logic" paper. He cites some simple schemas like '$A \rightarrow []A$' and '$][A \rightarrow A$' as examples of the kind of axioms that will probably be needed and announces that the compilation of a complete set of such axioms is "a major agendum." Some time ago I wrote a paper called "An Axiomatization of Predicate

Functor Logic" (henceforth "APL") in which I tried to give such an axiomatization. APL provides an explicit formal semantics for predicate functor logic and applies a Henkin-style argument to obtain completeness. I have never been entirely satisfied, however, with the axiomatization that results. The axioms use a special abbreviatory device, $\mathcal{T}$. For every sequence $k_1, \ldots, k_n$ of numbers, $\mathcal{T}\langle k_1, \ldots, k_n\rangle$ is a complicated string of functors. $\mathcal{T}\langle 3,4,2\rangle$, for example, contains more than thirty primitive functors. The axiom schemas Quine had suggested contain three or four functors each. My axiom schemas, on the other hand include schemas like "For all numbers n and all sequences of numbers $k_1, \ldots, k_n$, $\mathcal{T}\langle k_1 \ldots k_n\rangle A \rightarrow \mathcal{T}\langle k_2 \ldots k_n\rangle]A$." Quine's functors can be viewed as simple operations on sequences: "delete the first coordinate, "permute the first two coordinates," and so forth. His axioms show how these operations are interrelated. If our axioms were expressed in terms of the simple sequence-changing operations, they would be unintelligible.

For some time now I have been trying to derive my axioms from a set of simple ones, like those Quine suggested. I have come to the conclusion that some axioms much more complicated than Quine's will be indispensible. On the other hand, I am confident that an axiomatization can be given in which each axiom is simple enough to be intelligible without special definitions or abbreviations. This paper is written in order to report my progress on the problem and to solicit help in finding a solution.

The remainder of the paper is organized as follows. Section Two describes predicate functor logic and its interpretation and lists the axioms of APL. I have used the version of predicate functor logic with the reflection functor S and no identity predicate.

Section Three contains a new definition of τ which makes it easier to establish the relation between τ and the primitive functors Section Three also contains a new list of axioms which do not use the τ-functors, and a list of theorems which can be proved from the new axioms. These theorems facilitate the derivation in Section Four of the APL axioms.

II. Predicate Functor Logic

The language of predicate functor logic contains symbols of two varieties. First, for each $n \geq 0$ there is a countable collection of n-ary atomic predicates. For convenience we take these to be just the n-ary predicates of elementary logic. Second, there are the predicate functors, $-$, $\cap$, $\wp$, P, $[$, $]$, and S.

For $n \geq 0$ the set of n-ary predicates is the smallest set satisfying the following conditions:

1) All n-ary atomic predicates are n-ary predicates.
2) If A^n and B^m are n-ary and m-ary predicates, respectively, then $(A^n \cap B^m)$ is a max(m,n)-ary predicate.
3) If B^n is an n-ary predicate then $-B^n$, and PB^n are n-ary predicates.
4) If B^n is an n-ary predicate then $\wp B^n$ is an n-ary predicate unless $n < 2$, in which case $\wp B^n$ is a 2-ary predicate.
5) If B^n is an n-ary predicate then $[B^n$ is an n+1-ary predicate.

6) If B^n is an n-ary predicate, then $[B^n$ and $\mathcal{S}B^n$ are (n-1)-ary predicates unless n=0 in which case they are 0-ary predicates.

A predicate is a string of symbols which, for some n, can be shown to be an n-ary predicate on the basis of 1-6. (We make the usual assumptions that the initial collections of symbols are pairwise disjoint, and that juxtaposition in the metalanguage represents concatenation in the object language.) $\mathcal{L}_{PF}$ is the set of all predicates. A sentence of $\mathcal{L}_{PF}$ is a 0-ary predicate. Henceforth we use A, B, C and A^n, B^n, C^n, as metamathematical variables ranging over predicates in $\mathcal{L}_{PF}$ and n-ary predicates in $\mathcal{L}_{PF}$, respectively.

A model is a pair $M = \langle\mathcal{D},\mathcal{I}\rangle$ where $\mathcal{D}$ is a non-empty set (the domain of M) and $\mathcal{I}$ is a function from n-ary atomic predicates of $\mathcal{L}_{PF}$ to subsets of $\mathcal{D}^n$. The members of $\mathcal{D}^\omega$ are called arrays of individuals (on M), or simply arrays. Suppose $M = \langle\mathcal{D},\mathcal{I}\rangle$ is a model and $a = \langle d_1,d_2,\ldots\rangle$ is an individual array on M. Then B is true of a in M (written '$M \overset{a}{\models} B$' or simply '$a \models B$' when confusion is unlikely) if one of the following holds:

1) B is an atomic n-ary predicate and $\langle d_1,\ldots,d_n\rangle \in \mathcal{I}(B)$.

2) $B = -C$ and not $a \models C$.

3) $B = C \cap A$, and both $a \models C$ and $a \models A$.

4) $B = \mathfrak{p}C^n$ and $\langle d_2,d_1,d_3,d_4,\ldots\rangle \models C^n$.

5) $B = \mathcal{P}C^n$ and $\langle d_n,d_1,\ldots,d_{n-1},d_{n+1},\ldots\rangle \models C^n$.

6) $B = \mathcal{S}C^n$ and $\langle d_1,d_1,d_2,\ldots,d_n\rangle \models C^n$.

7) $B = [C$ and $\langle d_2,d_3,\ldots\rangle \models C$.

8) $B =]C$ and $\langle d_0,d_1,\ldots\rangle \models C$ for some $d_0 \in \mathcal{D}$.

P is true in M (written '$M \models P$') if $M \overset{a}{\models} P$ for all individual arrays $\underset{\sim}{a}$ on M. P is valid ('$\models P$') if it is true in all models. If $\Gamma \subseteq L_{PF}$ then Γ is true of a in M ('$M \overset{a}{\models} \Gamma$') if, for all $P \in \Gamma$, $M \overset{a}{\models} P$.

In APL the language was supplemented by a number of defined functors including, for every pair (m,n) of natural numbers a functor $\iota_{m,n}$ and, for every length-n sequence $<k_1,\ldots,k_n>$ of natural numbers, a functor $\tau<k_1,\ldots,k_n>$ satisfying the following.

ι-property

$$1 \leq m \leq p,\ \ 1 \leq n \leq p \Rightarrow <a_1,\ldots,a_p> \models \iota_{m,n}A^p \ \text{ iff}$$
$$<a_1,\ldots,a_{m-1},a_n,a_{m+1},\ldots,a_p> \models A^p \ .$$

τ-property

$$p \geq \max(k_1,\ldots,k_n) \Rightarrow <a_1,\ldots,a_p> \models \tau<k_1,\ldots,k_n> A^n \ \text{ iff}$$
$$<a_{k_1},a_{k_2},\ldots,a_{k_n}> \models A.$$

The class of valid predicates was shown to axiom tized by the following schemas and rules.

P1. All "tautologous" predicates, i.e. all predicates that can be obtained from tautologies of the propositional calculus by a uniform substitution of predicates for sentence letters, $-$ for $\sim$ and $\cap$ for $\wedge$.

P2. $\tau<1,\ldots,n>A^n \equiv A^n$.

P3. $\tau<k_1,\ldots,k_n>A^n \equiv \tau<k_1,\ldots,k_{n+p}>A^n$.

P4. $\tau<k_1,\ldots,k_n>-A \equiv -\tau<k_1,\ldots,k_n>A$

In addition the "boolean" functors and were added with appropriate definitions.

P5. $\tau<k_1,\ldots,k_n>(A \cap B) \equiv \tau<k_1,\ldots,k_n>A \cap \tau<k_1,\ldots,k_n>B$

P6. $\iota_{m,n}\iota_{n,m}A \equiv \iota_{m,n}A$

P7. $\iota_{mn}\iota_{pm}A \equiv \iota_{m,n}\iota_{p,n}A$

P8. $\tau K\iota_{m,n}A \equiv \tau K^1 A$ where K^1 results from replacing k_m in K by k_n.

P9. $\tau<k_1,\ldots,k_n>PA^n \equiv \tau<k_n,k_1,\ldots,k_{n-1}>A^n$

P10. $\tau<k_1,\ldots,k_n>pA^n \equiv \tau<k_2,k_1,k_3,\ldots,k_n>A^n$

P11. $\tau<k_1,\ldots,k_n>[A^{n-1} \equiv \tau<k_2,\ldots,k_n>A^{n-1}$

P12. $\tau<k_o,\ldots,k_n>A^{n+1} \rightarrow \tau<k_1,\ldots,k_n>]A^{n+1}$

PR1) $\vdash A, \vdash A \rightarrow B \Rightarrow \vdash B$

PR2) $\vdash A \Rightarrow \vdash \bigcirc A$ where $\bigcirc = p, P,], [, S$

PR3) $\vdash (A^a \cap \tau<k_1,\ldots,k_n>B^b) \rightarrow C^c \Rightarrow -(A^a \cap \tau<k_o,\ldots,k_n>]B^b) \rightarrow C^c$

(provided $k_o > \max(a,b,c,n,k_1,\ldots,k_n)$.)

III. The New Axioms

Defined Functors

Superscripts on functors or bracketed groups of functors indicate iteration. For example '$(Pp)^2A$' means $PpPpA$ and '$[^0A$' means A.

1. $A \rightarrow B = -(A \cap -B)$

2. $A \equiv B = (A \rightarrow B) \cap (B \rightarrow A)$

3. $\perp^0 = (P \cap -P)$ where P is the first 0-ary atomic predicate in some enumeration

4. $\perp^k = [^k\perp^0$

5. $\top^k = -\perp^k$

6. $P^{-1}A^n = \cancel{P^{n-1}A^n}$ $P^{-k}A^n = (P^{-1})^k A^n$

7. $p_k A^n = P^{1-k} p P^{k-1} \quad (1 \leq k < n)$

8. $p_k A^n = p_k (A \cap T^{k+1}) \quad (k \geq n)$

9. $$P_{j,k} A = \begin{cases} p_{j-1}\, p_{j-2} \cdots\cdots p_k\ A & (1 \leq k < j) \\ p_j\, p_{j+1} \cdots p_{k-1}\ A & (1 \leq j < k) \\ A & (j=k) \end{cases}$$

10. $P_k A = P_{k,1} A$

11. $P_k^{-1} A = P_{1,k} A$

$P_k^{-n} A = (P_k^{-1})^n A$

12. $iA = [SA$

13. $i_m A = P^{1-k} i P^{n-1} A$

14. $$i_{a,b} A = \begin{cases} P_{a,b-1}\ i_{b-1}\ P_{b,a} & (a < b) \\ P_{a,b}\ i_b\ P_{b,a} & (b < a) \\ A & (a=b) \end{cases}$$

Let $M = (\mathcal{D}, \mathcal{I})$ be a model and let $D = (d_1, d_2, \ldots.)$ be an array in M. It is easy to verify that the defined terms have the following properties.

a) D satisfies $p_k A$ if the sequence obtained by switching d_k and d_{k+1} in D satisfies A.

b) D satisfies $P_{k,m} A$ if the sequence obtained by removing d_k from D and reinserting it between d_{m-1} and d_m (so that it now occupies the m^{th} position) satisfies A.

c) D satisfies $i_k A$ if the result of changing d_k to d_{k+1} in D satisfies A.

d) D satisfies $i_{k,m} A$ if the result of changing d_k to d_m in D satisfies A.

Notation

It will be useful to have some notation for sequences of natural numbers. Unless otherwise indicated lower case plain face letters will stand for natural numbers and bold face $\mathbf{K}$ and $\mathbf{H}$ will stand for sequences of natural numbers. When sequences are arranged vertically a solid line will indicate that the coordinates not displayed are the same as the corresponding coordinates of the sequence immediately above. Dots are used in the usual way. For example, instead of

$$A = \langle k_1,\ldots,k_j,\ldots,k_n,\ldots,k_p\rangle \quad \text{and}$$

$$B = \langle k_1,\ldots,k_{j-1},k_n,k_{j+1},\ldots,k_{n-1},k_j,k_{n+1},\ldots,k_p\rangle,$$

we write

$$A = \langle k_1,\ldots,k_j,\ldots,k_n,\ldots,k_p\rangle,$$

$$B = \langle k_1 \text{———} \; k_n \text{———} \; k_j \text{———} \; k_p\rangle \; .$$

The definition of τ will require a few preliminaries.

Let $\mathbf{K} = (k_1,\ldots,k_p)$ and let $1 \leq i \leq p$, $1 \leq j \leq p$, $i \neq j$. If $k_i > k_{i+1}$, i is an <u>inversion</u> in $\mathbf{K}$. If $k_i = k_j$ then i is a <u>twin of j</u> in $\mathbf{K}$ and i is a <u>twin</u> in $\mathbf{K}$. If $a \leq \max(p_1,k_1,\ldots,k_p)$ and $a \neq k_m$ for $1 \leq m \leq p$ then a is an <u>absentee</u> of $\mathbf{K}$. Note that if $\mathbf{K}$ has a twin it must have an absentee.

We now define an ordering relation on sequences. Let $\mathbf{K} = (k_1,\ldots,k_p)$ and $\mathbf{K}^1 = (k_1^1,\ldots,k_q^1)$. $\mathbf{K} < \mathbf{K}^1$ iff

1) $\mathbf{K}$ has fewer absentees than $\mathbf{K}^1$ or
2) $\mathbf{K}$ and $\mathbf{K}^1$ have the same number of absentees but $\mathbf{K}^1$ has a twin greater than any twin of $\mathbf{K}$ or

3) K and K^1 have the same number of absentees and K^1 has no twin greater than all the twins of K and $k_1 < k_1^1$, or

4) K and K^1 have the same number of absentees and K^1 has no twin greater than all the twins of K and $k_1 < k_1^1$.

$$K > H \text{ iff } H < K.$$

For example

$$\langle 2,4,1,6,8,8,4\rangle >$$
$$\langle 2,4,5,4,7,7,1\rangle >$$
$$\langle 2,4,5,4,8,3,1\rangle >$$
$$\langle 2,4,5,6,8,3,1\rangle >$$
$$\langle 2,4,5,6,3,8,1\rangle >$$
$$\langle 2,3,4,5,6,8,1\rangle >$$
$$\langle 1,2,3,4,5,6,8\rangle >$$
$$\langle 1,2,3,4,5,6,7,8\rangle .$$

Let $K = (k_1,\ldots,k_p)$. Using the ordering relation just defined we define a complex functor τK.

1. $K = \langle 1,\ldots,p\rangle$. τK is the empty string of functors (i.e., $\tau K\, A=A$).
2. There are twins in K of which j is the greatest, and m is the greatest twin of j in k. Then

$$\tau\langle k_1,\ldots,k_m,\ldots,k_j,\ldots,k_p\rangle =$$
$$\tau\langle k_1 \text{——————} a \text{———} k_p\rangle \iota_{j,m}$$

where a is the smallest absentee of K.

3. There are no twins in K but there are absentees, of which a is the smallest. $\tau\langle k_1,\ldots,k_p\rangle = \tau\langle a,k_1,\ldots,k_p\rangle[$.
4. There are no twins or absentees in K but there are inversions of which j is the smallest. Then

$$\tau\langle k_1, \ldots, k_j, k_{j+1} \ldots, k_p\rangle =$$
$$\tau\langle k_1 \text{ —— } k_{j+1}, k_j \text{ — } k_p\rangle\, p_j .$$

It is easy to see that this definition assigns a unique string of functors to every sequence of numbers. It is also easy to see that this definition satisfies the τ-property. It remains to check that we can prove P1-P11 from simple axioms.

<u>Axioms</u>

Ax1. All tantologous functors

Ax2. $-\bigcirc A \equiv \bigcirc - A \qquad \bigcirc = p, P, [, S$

Ax3. $\bigcirc(A \cap B) \equiv (\bigcirc A \cap \bigcirc B) \qquad \bigcirc = p, [, S,]$

Ax3a. $P(A^n \cap B^n) \equiv PA^n \cap PB^n$

Ax4. $P^n A^n \equiv A^n$

Ax5. $\not{p}P^n\not{p}P^{-n}A^p \equiv P^n\not{p}P^{-n}\not{p}A^p \qquad (1 < n < p-1)$

Ax6. $\not{p}P\not{p}P^{-1}A \equiv P\not{p}P^{-1}\not{p}P\not{p}A$

Ax7. $\not{p}pA \equiv A$

Ax8. $SP^n\not{p}A^p \equiv P^n\not{p}^{-n}SP^nA$

Ax9. $SP\not{p}A \equiv P\not{p}SP^{-1}\not{p}PA$

Ax10. $SP^{-1}\not{p}A \equiv \not{p}P^{-1}SP\not{p}P^{-1}A$

Ax11. $S\not{p}A \equiv SA$

Ax12. $[\not{p}A \equiv P^{-1}\not{p}P[A$

Ax13. $[PA \equiv Pp[A$

Ax14. $[P^{-1}A \equiv pP^{-1}[\,A$

Ax15. $(Pp)^{\mathrm{P}-1}A \equiv A$

Ax16. $pA \equiv\ ]iP^{-1}ipPpipP^{-1}pP[A$

Ax17. $[i_{m,n}A \equiv i_{m+1,n+1}[A$

Ax18. $[A \equiv i_{1,m}[A$

Ax19. $[p,A \equiv p_{j+1}[A$

Ax20. $i_{m,n}i_{n,m}A \equiv i_{m,n}A$

Ax21. $i_{m,n}i_{a,m}A \equiv i_{a,n}i_{m,n}A$

Ax22. $i_{m,a}i_{m,b}A \equiv i_{m,b}A$

Ax23. $i_{a,b}i_{c,d}A \equiv i_{c,d}i_{a,b}A$ (where $a \neq c$, $a \neq d$, $b \neq c$)

Ax24. $p[A \equiv i_{2,1}[A$

Ax25. $p[[A \equiv [[A$

Ax26. $n > p \Rightarrow P_n[A^p \equiv A^p$

Ax27. $i_{m,n}]A \equiv\ ]i_{m+1,n+1}A$

Ax28. $p_j]A \equiv\ ]p_{j+1}A$

Ax29. $[PA \equiv Pp[A$

Ax30. $]p[A \equiv []A$

Ax31. $][A \equiv A$

Ax32. $A \equiv P[A$

Ax33. $PA^n \equiv A$

Ax34. $A \rightarrow []A$

Ax35. $S[A \equiv A$

R1. $\vdash A \Rightarrow \vdash B$ where B is a tautologous consequence of A

R2. $\vdash A \Rightarrow \vdash \bigcirc A \quad \bigcirc = p, P, [, i,]$

R3. $m > n, \vdash P_m A^n \Rightarrow \vdash -]-A$

Theorems

T1. $\vdash \bigcirc A \equiv \bigcirc^1 A \Rightarrow \vdash B \equiv B^1$ where $\bigcirc$ and $\bigcirc^1$ are strings of functors and B^1 results from B by replacing one or more occurances of $\bigcirc$ by $\bigcirc^1$.

T2. $P^{m+p} A^p \equiv P^m A$

T3. $|m-n| \geq 2 \Rightarrow p_m p_n A \equiv p_n p_m A$

T4. $p_m p_{m+1} p_m A \equiv p_{m+1} p_m p_{m+1} A$

T5. $p_m p_m A \equiv A$

T6. $1 < n < p-1 \Rightarrow i P^n p A \equiv (Pp)^n P^{-1} (p P^{-1})^n i P^n A$

T7. $i P p A \equiv P p P^{-1} p P i P^{-1} p P A$

T8. $i P^{-1} p A \equiv P^{-1} p P p P^{-1} i P p P^{-1} A$

T9. $i p A \equiv i A$

T11. a) $m < n$

i. $k < m-1 \Rightarrow P_{m,n} p_k A \equiv p_k P_{m,n} A$

ii. $k = m-1 \Rightarrow P_{m,n} p_k A \equiv p_k p_m P_{k,n} A$

iii. $m \leq k < n-1 \Rightarrow P_{m,n} p_k A \equiv p_{k+1} P_{m,n} A$

iv. $k = n-1 \Rightarrow P_{m,n} p_k A \equiv P_{m,n-1} A$

v. $k = n \Rightarrow P_{m,n} p_k A \equiv P_{m,n+1} A$

vi. $k > n \Rightarrow P_{m,n} p_n A \equiv p_k P_{m,n} A$

T11. b) $n < m$

i. $k < n-1 \Rightarrow P_{m,n}p_kA \equiv p_kP_{m,n}A$

ii. $k = n-1 \Rightarrow P_{m,n}p_kA \equiv P_{m,n-1}A$

iii. $k = n \Rightarrow P_{m,n}p_kA \equiv P_{m,n+1}A$

iv. $n < k \leq m-1 \Rightarrow P_{m,n}p_kA \equiv p_{k-1}P_{m,n}A$

v. $k = m \Rightarrow P_{m,n}p_kA \equiv p_kp_{k-1}P_{m+1,n}A$

vi. $k > m \Rightarrow P_{m,n}p_kA \equiv p_kP_{m,n}A$

T10. $p_mp_{m,n}A \equiv P_{m+1,n}A$

$p_{m-1}p_{m,n}A \equiv P_{m-1,n}A$

T12. i. $j < k-1 \Rightarrow i_jp_kA \equiv p_ki_jA$

ii. $j = k-1 \Rightarrow i_jp_kA \equiv p_kp_ji_kp_j$

iii. $j = k \Rightarrow i_jp_kA \equiv i_jA$

iv. $j = k+1 \Rightarrow i_jp_kA \equiv p_kp_ji_kp_jA$

v. $j > k+1 \Rightarrow i_jp_kA \equiv p_ki_jA$

T13. a) $m < n-1$

$j < m-1 \Rightarrow i_{m,n}p_jA \equiv p_ji_{m,n}A$

$j = m-1 \Rightarrow i_{m,n}p_jA \equiv p_ji_{j,n}A$

$j = m \Rightarrow i_{m,n}p_jA \equiv p_ji_{j+1,n}A$

$m < j < n-1 \Rightarrow i_{m,n}p_jA \equiv p_ji_{m,n}A$

$j = n-1 \Rightarrow i_{m,n}p_jA \equiv p_ji_{m,n-1}A$

$j = n \Rightarrow i_{m,n}p_jA \equiv p_ji_{m,n+1}A$

$j > n \Rightarrow i_{m,n}p_jA \equiv p_ji_{m,n}A$

T13. b) $m = n-1$

$j < m-1 \Rightarrow \imath_{m,n} p_j A \equiv p_j \imath_{m,n} A$

$j = m-1 \Rightarrow \imath_{m,n} p_j A \equiv p_j \imath_{j,n} A$

$j = m = n-1 \Rightarrow \imath_{m,n} p_j A \equiv \imath_m A$

$j = n \Rightarrow \imath_{m,n} p_j A \equiv p_j \imath_{m,n+1} A$

$j > n \Rightarrow \imath_{m,n} p_j A \equiv p_j \imath_{m,n} A$

c) $n = m-1$

$j < n-1 \Rightarrow \imath_{m,n} p_j A \equiv p_j \imath_{m,n} A$

$j = n-1 \Rightarrow \imath_{m,n} p_j A \equiv p_j \imath_{m,j} A$

$j = n = m-1 \Rightarrow \imath_{m,n} p_j A \equiv \imath_m A$

$j = m \Rightarrow \imath_{m,n} p_j A \equiv p_j \imath_{m+1,n} A$

$j > m \Rightarrow \imath_{m,n} p_j A \equiv p_j \imath_{m,n} A$

d) $n < m-1$

$j < n \Rightarrow \imath_{m,n} p_j A \equiv p_j \imath_{m,n} A$

$j = n-1 \Rightarrow \imath_{m,n} p_j A \equiv p_j \imath_{m,j} A$

$j = n \Rightarrow \imath_{m,n} p_j A \equiv p_j \imath_{m+1,j} A$

$n < j < m-1 \Rightarrow \imath_{m,n} p_j A \equiv p_j \imath_{m,n} A$

$j = m-1 \Rightarrow \imath_{m,n} p_j A \equiv p_j \imath_{m-1,n} A$

$j = m \Rightarrow \imath_{m,n} p_j A \equiv p_j \imath_{m+1,n} A$

$j > m \Rightarrow \imath_{m,n} p_j A \equiv p_j \imath_{m,n} A$

T14. $(p P^{-1})^{-n} A \equiv (P p)^{n-1} A$

T15. $p_m P^{-1} A^p \equiv P^{-1} p_{m-1} A^p \qquad 1 < m < p$

$p P^{-1} A^p \equiv P^{-2} p_{p-1} P A^p$

T16. $p_m P A^p \equiv P p_{m+1} A^p \qquad 1 \leq m < p-1$

$p_m P A^p \equiv P^2 p P^{-1} A^p \qquad m = p-1$

T17. $\mathcal{P}\mathcal{P}_{m,n}A \equiv \mathcal{P}_{m-1,n-1}A \quad m,n > 1$

$\mathcal{P}\mathcal{P}_{1,n}A^{p} \equiv \mathcal{P}_{p,n}A^{p}$

$\mathcal{P}\mathcal{P}_{1,m}A^{p} \equiv \mathcal{P}_{m-1,p-1}\mathcal{P}^{2}A^{p}$

T18. $\mathcal{P}i_{k}A \equiv i_{k-1}\mathcal{P}A \quad 1 < k \leq P$

$\mathcal{P}iA^{p} \equiv \mathcal{P}^{-1}i_{p-1}\mathcal{P}^{2}A$

T19. $\mathcal{P}i_{k,n}A \equiv i_{k-1,n-1}A \quad 1 < n,k$

$\mathcal{P}i_{k,1}A^{p} \equiv i_{k-1,p}A \quad 1 < k$

T20. $\mathcal{P}^{k-1}i_{k,n}A \equiv i_{1,(n-k)+1}\mathcal{P}^{n-1}A$

T21. $p_{k}i_{k,n}A \equiv \mathcal{P}^{2-k}p\,i_{1,(n-k)+1}\mathcal{P}^{k-1}A$

T22. $pA \equiv]i_{1,2}i_{2,3}i_{3,1}[A$

T23. $i_{m,n}i_{m,n}A \equiv i_{m,n}A$

T24. $i_{m,n}SA \equiv i_{m+1,n+1}A$

T25. $p\,i_{1,n}A \equiv i_{2,1}i_{1,n}A$

T26. $p_{k}i_{k,m}i_{k+1,n}A \equiv i_{k,m}i_{k+1,n}A$

T27. $k > m,\ k > n \Rightarrow \ \vdash \mathcal{P}_{k}i_{m,n}A \equiv i_{m-1,n-1}\mathcal{P}_{k}A$

T28. $[\mathcal{P}_{m}A \equiv \mathcal{P}_{m}p[A$

T29. $\mathcal{P}A^{n} \equiv \mathcal{P}_{n}A$

T30. $k \geq n \Rightarrow A^{n} \equiv \mathcal{P}_{k}[A$

T31. $-]-[A \equiv A$

Proofs of selected theorems

T3

$$
\begin{aligned}
\wp_j \wp_k A &= P^{(p-j)+1} \wp P^{j-1} P^{1-k} \wp P^{k-1} A \\
&= P^{1-j} \wp P^{j-k} \wp P^{k-1} A \\
&= P^{1-j} \wp P^{j-k} \wp P^{k-j} P^{j-1} A \\
&\equiv P^{1-j} \mathbf{P}^{j-k} \wp P^{k-j} \wp P^{j-1} A \qquad \text{(by Ax5)} \\
&= P^{1-k} \wp P^{k-1} P^{1-j} \wp P^{j-1} A \\
&= \wp_k \wp_j A \ .
\end{aligned}
$$

T7

$$
\begin{aligned}
i P \wp A &= [S P \wp A \\
&\equiv [P \wp S P^{-1} \wp P A \qquad \text{(by Ax9)} \\
&\equiv P \wp [\wp S P^{-1} \wp P A \qquad \text{(by Ax13)} \\
&\equiv P \wp P^{-1} \wp P [S P^{-1} \wp P A \\
&\equiv P \wp P^{-1} \wp P i P^{-1} \wp P A \ .
\end{aligned}
$$

T11a

i) $k < m-1$

$P_{m,n} \wp_k A =$

$\wp_m \wp_{n+1} \cdots \wp_{n-1} \wp_n A \equiv$

$\wp_k \wp_m \wp_{m+1} \cdots \wp_{n-1} A$ (by repeated application of T3) $=$

$\wp_k P_{m,n} A$.

ii) $k = m-1$

$$\mathcal{P}_{m,n}p_k A = p_m p_{m+1} \cdots p_{n-1}p_k A$$

$$\equiv p_m p_k p_{m+1} \cdots p_{n-1} A \qquad \text{(by T3)}$$

$$\equiv p_m p_k p_m p_m p_{m+1} \cdots p_{n-1} A \qquad \text{(by T5)}$$

$$\equiv p_k p_m p_k p_m p_{m+1} \cdots p_{n-1} A \qquad \text{(by T4)}$$

$$= p_k p_m \mathcal{P}_{k,n-1} A \ .$$

iii) $m \leq k \leq n-1$

$$\mathcal{P}_{m,n}p_k A = p_m p_{m+1} \cdots p_n p_k A$$

$$= p_m \cdots p_k p_{k+1} \cdots p_{n-1} A$$

$$\equiv p_m \cdots p_k p_{k+1} p_k p_{k+2} \cdots p_{n-1} A$$

$$\equiv p_m \cdots p_{k-1} p_{k+1} p_k p_{k+1} p_{k+2} \cdots p_{n-1} A$$

$$\equiv p_{k+1} \mathcal{F}_{m,n} A \ .$$

iv) $k = n-1$

$$\mathcal{P}_{m,n}p_k A =$$

$$p_m p_{m+1} \cdots p_{n-1} p_k A \equiv$$

$$p_m p_{m+1} \cdots p_{n-2} A \qquad \text{(by T5)} =$$

$$\mathcal{P}_{m,n-1} A \ .$$

v) $k = n$

$P_{m,n}p_k A =$

$p_m p_{m+1} \cdots p_{n-1,n} A =$

$P_{m,n+1} A$

vi) $k > n$. Similar to case i.

<u>T13a</u>

i) $j < m-1$.

$i_{m,n} p_j A =$

$P_{m,n-1} i_{n-1} P_{n,m} p_j A \equiv$

$P_{m,n-1} i_{n-1} p_j P_{n,m} A$ (by T11) $\equiv$

$P_{m,n-1} p_j i_{n-1} P_{n,m} A$ (by T12) $\equiv$

$p_j P_{m,n-1} i_{n-1} P_{n,m} A$ (by T11).

ii) $j = m-1$.

$i_{m,n} p_j A =$

$P_{m,n-1} i_{n-1} P_{n,m} p_{m-1} A \equiv$

$P_{m,n-1} i_{n-1} P_{n,m-1} A$ $\equiv$

$p_{m-1} P_{m-1,n-1} i_{n-1} P_{n,m-1} A =$

$p_j P_{m-1,n} A$.

iii) $j = m$.

$$i_{m,n} p_j A =$$

$$P_{m,n-1} i_{n-1} P_{n,m} p_m A \equiv$$

$$P_{m,n-1} i_{n-1} P_{n,m+1} A \equiv$$

$$p_m P_{m+1,n-1} i_{n-1} P_{n,m+1} A \equiv$$

$$p_j i_{m+1,n} A \ .$$

iv) $m < j < n-1$. Similar to case i)

v) $j = n-1$.

$$i_{m,n} p_j A =$$

$$P_{m,n-1} i_{n-1} P_{n,m} p_{n-1} A \equiv$$

$$P_{m,n-1} i_{n-1} p_{n-2} P_{n,m} A \equiv$$

$$P_{m,n-1} p_{n-2} p_{n-1} i_{n-2} p_{n-1} P_{n,m} A \equiv$$

$$P_{m,n-2} p_{n-1} i_{n-2} P_{n-1,m} A \equiv$$

$$p_{n-1} P_{m,n-2} i_{n-2} P_{n-1,m} A =$$

$$p_j P_{m,n-1} A \ .$$

vi) $j = n$.

$$i_{m,n} P_j A =$$

$$P_{m,n-1} i_{n-1} P_{n,m} p_n A \equiv$$

$$P_{m,n-1} i_{n-1} p_n p_{n-1} P_{n+1,m} A \equiv$$

$$P_{m,n-1} p_n p_{n-1} i_n p_{n-1} p_{n-1} P_{n+1,m} A \equiv$$

$$P_{m,n-1}p_n p_{n-1} i_n P_{n+1,m} A \equiv$$

$$P_n P_{m,n-1} p_{n-1} i_n P_{n+1,m} A \equiv$$

$$p_n P_{m,n} i_n P_{n+1,m} A \equiv$$

$$p_j i_{m,n+1} A \ .$$

vii) Similar to case 1.

<u>T16</u>

i) $1 \leq m < p-1$.

$$p_m PA =$$

$$P^{1-m} p P^{m-1} PA =$$

$$PP^{-m} p P^{-m} A =$$

$$P p_{m+1} A \ .$$

ii) $m = p-1$

$$p_m PA =$$

$$P^2 p P^{-2} PA =$$

$$P^2 p P^{-1} A \ .$$

<u>T25</u>

$$p i_{1,n} A \equiv \,]i_{1,2} i_{2,3} i_{3,1} [i_{1,n} A \qquad \text{(by T22)}$$

$$\equiv \,]i_{1,2} i_{2,3} i_{3,1} i_{2,n+1} [A \qquad \text{(by Ax17)}$$

$$\equiv \,]i_{1,2} i_{2,3} i_{2,n+1} i_{3,1} [A \qquad \text{(by Ax23)}$$

$$\equiv \,]i_{1,2} i_{2,n+1} i_{3,1} [A \qquad \text{(by Ax22)}$$

$$\equiv\ ]i_{1,2}i_{3,1}i_{2,n+1}[A \qquad \text{(by Ax23)}$$

$$\equiv\ ]i_{1,2}i_{3,2}i_{2,n+1}[A \qquad \text{(by Ax21)}$$

$$=\ ][Si_{3,2}i_{2,n+1}[A$$

$$\equiv\ Si_{3,2}i_{2,n+1}[A \qquad \text{(by Ax31)}$$

$$\equiv\ i_{2,1}i_{1,n}S[A \qquad \text{(by T24)}$$

$$\equiv\ i_{2,1}i_{1,n}A \qquad \text{(by Ax35)} .$$

<u>T26</u>

$$p_k i_{k,m}i_{k+1,n}A =$$

$$P^{1-k}pP^{k-1}i_{k,m}i_{k+1,n}A \equiv$$

$$P^{1-k}pi_{1,(m-k)+1}P^{k-1}i_{k+1,n}A \qquad \text{(by T20)} \equiv$$

$$P^{1-k}i_{2,1}i_{1,(m-k)+1}P^{k-1}i_{k+1,m}A \qquad \text{(by T24)} \equiv$$

$$P^{1-k}i_{2,1}i_{1,(m-k)+1}i_{2,(n-k)+1},P^{k-1}A \qquad \text{(by T19)} \equiv$$

$$P^{1-k}i_{2,1}i_{2,(n-k)+1}i_{1,(m-k)+1}P^{k-1}A \qquad \text{(by Ax23)} \equiv$$

$$P^{1-k}i_{2,(n-k)+1}i_{1,(m-k)+1}A \qquad \text{(by Ax22)} \equiv$$

$$i_{k+1,n}i_{k,m}A \qquad \text{(by T19)} \equiv$$

$$i_{k,m}i_{k+1,n}A .$$

<u>T29</u>

$$P_nA^n = p_{n-1} \cdots p_1A^n$$

$$= P^{2-n}pP^{n-2}p^{3-n}pP^{n-3} \ \ldots\ P^{n-n}pP^{n-n}A^n$$

$$= P^{1-n}(P\rho)^{n-1}A^n$$

$$\equiv PA^n$$

T30

$$P_k[A^n = P_k([A^n \cap T^{k-1})$$

$$\equiv P([A^n \cap T^{k-1}) \qquad \text{(by L2a)}$$

$$\equiv P[(A^n \cap T^{k-2}) \qquad \text{(by Ax3)}$$

$$\equiv A^n \cap T^{k-2} \qquad \text{(by Ax32)}$$

$$\equiv A^n \; .$$

IV Derivation of the APL Axioms

Lemma 1. Let $K = \langle k_1, \ldots, k_j, \ldots, k_p \rangle$ where $k_j = k_{j+1}$. Then $\tau K \iota_{m,j} A \equiv \tau K \iota_{m,j+1} A$.

Proof. $\tau K\, \iota_{m,j} = \tau K^1\, \iota_{j+1,j} \iota_{a_1,b_1} \cdots \iota_{a_n,b_n} \iota_{m,j}$

where $j+1 < a_1 < a_2 < \cdots < a_n$ and $b_i < a_i$. By Ax22 and Ax19 this is equivalent to $\tau K^1 \iota_{j+1,j}\; \iota_{a_1,b_1} \cdots \iota_{a_n,b_n} \iota_{m,j+1}$ i.e., to $\tau K^1 \iota_{m,j+1}$.

Lemma 2. Let $K = \langle k_1, \ldots\ldots, k_j, \ldots\ldots k_p \rangle$ and

$$K^1 = \langle k_1 \text{ — } k_{j+1}, k_j \text{ — } k_p \rangle.$$

Then $\tau K \rho_j A \equiv \tau K^1 A$.

Proof. Induction on K

Case 1. $K = \langle 1, \ldots, p \rangle$. Then $\tau K A = \tau K^1 A$.

Case 2. $\mathcal{K}$ has twins, the greatest of which is n and the greatest twin of n in $\mathcal{K}$ is m.

a) $1 \leq j \leq m-1$. $\tau\mathcal{K}\rho_j A =$

$$\tau<k_1, \ldots, k_j, k_{j+1}, \ldots, k_m, \ldots, k_n, \ldots, k_p>\rho_j A \equiv$$

$$\tau<k_1 \text{———————} a \text{———} k_p>\iota_{n,m}\rho_j A \equiv$$

(where a is the smallest absentee in $\mathcal{K}$)

$$\tau<k_1 \text{————————————} k_p>\rho_j\iota_{n,m} A$$

(by L13) $\equiv$

$$\tau<k_1 \text{———} k_{j+1}, k_j, \text{———————} k_p>\iota_{n,m}$$

(by induction hypothesis) $=$

$$\tau\mathcal{K}^1 A$$

(since the absentees of $\mathcal{K}$ and $\mathcal{K}^1$ are the same).

b) $m < j < n-1$ or $n < i \leq p$. Similar to the previous case.

c) $j=m-1$ and j is not a twin of n.

$$\tau\mathcal{K}\rho_j A =$$

$$\tau<k_1, \ldots, k_{m-1}, k_m, \ldots, k_n, \ldots, k_p>\rho_j A =$$

$$\tau<k_1 \text{——————} a \text{———} k_p>\iota_{n,m}\rho_j A \equiv$$

$$\tau<k_1 \text{—————————} k_p>\rho_j\iota_{n,m-1} A \equiv$$

$$\tau<k_1 \text{———} k_m, k_{m-1} \text{—————} k_p>\iota_{n,m-1} A =$$

$$\tau\mathcal{K}^1 A$$

d) $j = m-1$ and j is a twin of n

$\tau K^{1}\mathcal{A} =$

$\tau\langle k_1, \ldots, k_{m-2}, k_m, k_{m-1}, k_{m+1}, \ldots, k_n, \ldots, k_p\rangle A$

$= \tau\langle k_1 \text{ ————————————— } a \text{ ——— } k_p\rangle \iota_{n,m} A$

$\equiv \tau\langle k_1 \text{ ————— } k_{m-1}, k_m \text{ ————————— } k_p\rangle \rho_j \iota_{n,m} A$

$\equiv \tau\langle k_1 \text{ ——————————————————— } k_p\rangle \iota_{n,m-1} \rho_j A$

$\equiv \tau\langle k_1 \text{ ——————————————————— } k_p\rangle \iota_{n,m} \rho_j A$

(by Lemma 1)

$= \tau\langle k_1 \text{ ————————————— } k_n \text{ ——— } k_p\rangle \rho_j A$

e) $j = m$.

$\tau K^{1}\mathcal{A} =$

$\tau\langle k_1, \ldots, k_{m-1}, k_{m+1}, k_m, \ldots, k_n, \ldots, k_p\rangle A$

$= \tau\langle k_1 \text{ ——————————— } a \text{ ——— } k_p\rangle \iota_{n,m+1} A$

$\equiv \tau\langle k_1 \text{ ————— } k_m, k_{m+1} \text{ ————— } k_p\rangle \rho_i \iota_{n,m+1} A$

$\equiv \tau\langle k_1 \text{ ———————————————— } k_p\rangle \iota_{n,m} \rho_i \iota$

$= \tau\langle k_1 \text{ ——————————— } k_n \text{ ——— } k_p\rangle \rho_i A$

$= \tau K \rho_i A$

f) $j = n-1$ and $h-1$ is not a twin in $h, \tau_{K'}\mathcal{A} =$

$\tau\langle k_1, \ldots, k_m, \ldots, k_{n-2}, k_n, k_{n-1}, k_{n+1}, \ldots, k_p\rangle A$

$= \tau\langle k_1 \text{ ————————— } a \text{ ———————— } k_p\rangle \iota_{n-1,m} A$

$\equiv \tau\langle \text{ ————————— } k_{n-1}, a \text{ ——————— } k_p\rangle \rho_{n-1} \iota_{n-1,m} A$

$\equiv \tau<$ ——————————————————— $>\iota_{n,m}\rho_j A$

$\equiv \tau<$ ——————————— k_n ——————— $>\rho_j A$

$= \tau K \rho_j A$

g) $j = n-1$, $m < n-1$ and $n-1$ is a twin in K. Let q be the largest twin of $n-1$ in K. (Since m is the largest twin of n, $k_q \neq k_m$ and $q \neq m$.)

$\tau K_j A =$

$\tau<k_1,\ldots,k_q,\ldots,k_{n-2},k_{n-1},a,k_{n+1},\ldots,k_p>\iota_{n,m}\rho_j A \equiv$

$\tau<k_1$ ——————————————— $k_p>\rho_j\iota_{n-1,m}A \equiv$

$\tau<k_1$ ——————————— a,k_{n-1} —— $k_p>\iota_{n-1,m}A \equiv$

$\tau<k_1$ ——————————— a,b ——— $k_p>\iota_{n,q}\iota_{n-1,m}A$

(where b is the smallest absentee in the previous squence) $\equiv$

$\tau<k_1$ ——————————————— $k_p>\iota_{n-1,m}\iota_{n,q}A \equiv$

$\tau<k_1$ ——————————————— $k_p>\rho_{n-1}\iota_{n-1,m}\iota_{n,q}A$

(by L25) $\equiv$

$\tau<k_1$ ——————————— b,a ——— $k_p>\iota_{n-1,n}\iota_{n,q}A$

(by induction hypothesis) $=$

$\tau<k_1$ ——————————— k_n,a ——— $k_p>\iota_{n,q}A =$

$\tau<k_1$ ——————————— k_n,k_{n-1} ——— $k_p>A =$

$\tau K^1 A$

h) $j = n-1 = m$

$\tau K \wp_1 A =$

$\tau\langle k_1, \ldots, k_{n-1}, k_n, \ldots, k_p\rangle \wp_j A =$

$\tau\langle k_1$ ———— a ——— $k_p\rangle \iota_{n,n-1}\wp_{n-1}A \equiv$

$\tau\langle k_1$ ———————— $k_p\rangle \iota_{n,n-1}A =$

$\tau\langle k_1$ ———— k_{n-1} —— $k_p\rangle A =$

$\tau K^1 A$

i) $j = n$. $\tau K \wp_j A =$

$\tau\langle k_1, \ldots, k_m, \ldots, k_n, \ldots, k_p\rangle \wp_j A =$

$\tau\langle k_1, \ldots, k_m, \ldots, a, \ldots, k_p\rangle \iota_{n,m}\wp_n A \equiv$

$\tau\langle k_1$ ———————— $k_p\rangle \wp_n \iota_{n+1,m}A \equiv$

$\tau\langle k_1$ ———— k_{n-1}, a —— $k_p\rangle \iota_{n+1,m}A \equiv$

$\tau\langle k_1$ ———— k_n ——— $k_p\rangle A$

$= \tau k^1 A$

Case 3. There are no twins in K but there are absentees, of which a is the smallest.

$\tau\langle k_1, \ldots, k_p\rangle \wp_j A =$

$\tau\langle a, k_1,$ —— $k_p\rangle [\wp_j A$

$\equiv \tau\langle a, k_1,$ —— $k_p\rangle \wp_{j+1} [A$

$\equiv \tau\langle a, k_1,$ —— k_{j+1}, k_j —— $k_p\rangle [A$

$= \tau\langle k_1,$ —— k_{j+1}, k_j —— $k_p\rangle A$

Case 4. K has no twins or absentees but it does have inversions, of which m is the smallest.

a) $m < j-1$

$$\tau K \rho_j A =$$

$$\tau\langle k_1, \ldots, k_m, \ldots, k_j, \ldots, k_p \rangle \rho_j A$$

$$= \tau\langle k_1, \text{———} \; k_{m+1}, k_m \text{—————} k_p \rangle \rho_m \rho_j A$$

$$\equiv \tau\langle k_1 \text{——————————————} k_p \rangle \rho_j \rho_m A$$

$$\equiv \tau\langle k_1 \text{————————} k_{j-1} \; k_j \text{——} k_p \rangle \rho_m A$$

$$\equiv \tau\langle k_1 \text{———} k_m, k_{m+1} \text{—————} k_p \rangle A$$

$$= \tau K^1 A$$

b) $m = j-1$, $k_j < k_{j-1}$ and $k_{j+1} < k_{j-1}$.

$$\tau K \rho_j =$$

$$\tau\langle k_1, \ldots, k_{j-2}, k_{j-1}, k_j, k_{j+1}, \ldots k_p \rangle \rho_j A$$

$$= \tau\langle k_1 \text{———} k_j, k_{j-1} \text{——————————} k_p \rangle \rho_m \rho_j A$$

$$\equiv \tau\langle k_1 \text{———}, k_{j+1}, k_{j-1}, \text{————————} k_p \rangle \rho_j \rho_m \rho_j A$$

$$\equiv \tau\langle k_1 \text{————————————————————} k_p \rangle \rho_m \rho_j \rho_m A$$

$$\equiv \tau\langle k_1 \text{———} k_{j+1}, k_j, k_{j-1} \text{—————————} k_p \rangle \rho_j \rho_m A$$

$$\equiv \tau\langle k_1 \text{——————} k_{j-1}, k_j \text{————————} k_p \rangle \rho_m A$$

$$\equiv \tau\langle k_1 \text{———} k_{j-1}, k_{j+1}, k_j \text{—————————} k_p \rangle A$$

$$= \tau K^1 A$$

c) $m = j-1$ and $k_j < k_{j-1} < k_{j+1}$.

$$\tau \mathcal{K}^1 A =$$

$$\tau\langle k_1, \ldots, k_{j-1}, k_{j+1}, k_j, \ldots, k_p\rangle A$$

$$= \tau\langle k_1 \text{ ———————— }, k_j, k_{j+1}, \text{ — } k_p\rangle \wp_j A$$

$$= \tau \mathcal{K} \wp_j A$$

d) $m = j$

$$\tau \mathcal{K} \wp_j A =$$

$$\tau\langle k_1, \ldots, k_m, \ldots, k_p\rangle \wp_m A =$$

$$\tau\langle k_1 \text{ —}, k_{m+1}, k_m, \text{ — } k_p\rangle \wp_m \wp_m A$$

$$\equiv \tau\langle k_1 \text{ ———————————— } k_p\rangle A$$

$$= \tau \mathcal{K}^1 A$$

e) $m > j$. Since m is the smallest inversion, $k_j < k_{j+1}$.

$$\tau \mathcal{K}^1 A =$$

$$\tau\langle k_1, \ldots, k_{j-1}, k_{j+1}, k_j, k_{j+2}, \ldots, k_p\rangle A$$

$$= \tau\langle k_1, \text{ ——————— } k_j, k_{j+1} \text{ ——————— } k_p\rangle \wp_j A$$

$$= \tau \mathcal{K} \wp_j A$$

<u>Lemma 3</u>. Let $\mathcal{K} = \langle k_1, \ldots, k_p\rangle$, $1 \leq m \leq p$, $1 \leq n \leq p$ and let $\mathcal{K}^1$ be the result of replacing k_m in $\mathcal{K}^1$ by k_n (leaving all other coordinates unchanged). Then $\tau \mathcal{K} \iota_{m,n} A \equiv \tau \mathcal{K}^1 A$.

<u>Proof</u>. (Induction on $\mathcal{K}$.)

Case 1. $k = \langle 1, \ldots, p\rangle$. Then $\tau\mathcal{K}^1 A = \tau\mathcal{K} i_{m,n} A$.

Case 2. $\mathcal{K}$ contains twins, the largest of which is j and the largest twin of j in $\mathcal{K}$ is h.

a) $m = j$. Then $n \lesssim h$.

$$\tau_{\mathcal{K}} i_{m,n} A =$$

$$\tau\langle k_1, \ldots, k_n, \ldots, k_h, \ldots, k_j, \ldots, k_p\rangle i_{m,n} A$$

$$= \tau\langle k_1 \text{ ——————————— } a \text{ ——— } k_p\rangle i_{m,h} i_{m,n} A$$

$$\equiv \tau\langle k_1 \text{ ——————————————— } k_p\rangle i_{m,h} A$$

$$= \tau\langle k_1 \text{ ——————————— } k_j \text{ ——— } k_p\rangle A$$

$$= \tau\mathcal{K}^1 A$$

b) $n = j$, $m = h$.

$$\tau\mathcal{K} i_{m,n} A =$$

$$\tau\langle k_1, \ldots, k_h, \ldots, k_j, \ldots, k_p\rangle i_{m,h} A$$

$$= \tau\langle k_1 \text{ ——————— } a \text{ ——— } k_p\rangle i_{n,m} i_{m,n} A$$

$$\equiv \tau\langle k_1 \text{ ——————— } a \text{ ——— } k_p\rangle i_{n,m} A$$

$$= \tau\langle k_1 \text{ ——————— } k_j \text{ ——— } k_p\rangle A$$

$$= \tau\mathcal{K}^1 A$$

c) $n = j$, $m < h$. (Then $k_h = k_j = k_h$.)

$$\tau\mathcal{K} i_{m,n} A =$$

$$\tau\langle k_1 \text{ ——— } k_m \text{ ——— } k_h \text{ ——— } k_j \text{ ——— } k_p\rangle i_{m,n} A$$

$$= \tau\langle k_1 \text{ ——————————— } a \text{ ——— } k_p\rangle i_{n,h} i_{m,n} A$$

$\equiv \tau\langle k_1$ ———————————— a ——— $k_p\rangle \iota_{mh}\iota_{jh}A$

$\equiv \tau\langle k_1$ ——— k_h ——— k_h ——— a ——— $k_p\rangle \iota_{jh}A$

$= \tau\langle k_1$ ———————————— k_h ——— $k_p\rangle A$

$= \tau K^1 A$

d. $h < n < j$ or $h < m < j$. $k_m = k_n$, $k_h = k_1$ and h is the greatest twin of j, so $k_m \neq k_h$ and hence $m \neq h$, $n \neq h$, $m \neq j$, $n \neq j$.

$\tau K \iota_{mn} A =$

$\tau\langle k_1, \ldots, k_{j-1}, a, k_{j+1}, \ldots, k_p\rangle \iota_{j,h}\iota_{m,n}A \equiv$

$\tau\langle k_1$ ———————————— $k_p\rangle \iota_{m,n}\iota_{jh}A \equiv$

$\tau\langle k_1^1, \ldots, k_{j-1}^1, a, k_{j+1}^1, \ldots, k_p^1\rangle \iota_{jn}A =$

$\tau K^1 A$

e. $m = h$, $k_n \neq k_m$. Then $K^1 < K$. $\tau_K \equiv \tau_K{}^1\iota_{mj}$ by induction hypothesis. So

$$\tau K \iota_{mn} \equiv \tau_{K^1}\iota_{mj}\iota_{mn} A$$
$$\equiv \tau k^1 \iota_{mn} A$$
$$\equiv \tau K' A.$$

f. $m = h$, $k_n = k_m$.

$\tau K \iota_{mn} A =$

$\tau\langle k_1, \ldots, k_n, \ldots, k_m, \ldots, k_j, \ldots, k_p\rangle \iota_{mn} A =$

$\tau\langle k_1,$ ———————————— a ——— $k_p\rangle \iota_{j,m}\iota_{mn}A \equiv$

$$\tau < k_1 \text{ ——————————— } a \text{ ——— } k_p > \imath_{n,m}\, \imath_{j,m}\, \imath_{m,n} A$$

because $k_n = k_m$ and $< k_1 \text{ —— } a \text{ —— } k_p > \, < K$

$$\equiv \tau < k_1 \text{ ———————————————— } k_p > \imath_{j,m}\, \imath_{n,m}\, \imath_{m,n} A$$

$$\equiv \tau < k_1 \text{ ———————————————— } k_p > \imath_{j,m}\, \imath_{n,m} A$$

$$\equiv \tau < k_1 \text{ ———————————————— } k_p > \imath_{n,m}\, \imath_{j,m} A$$

$$\equiv \tau < k_1 \text{ ———————————————— } k_p > \imath_{j,m} A$$

$$= \tau K^1$$

g. $n = h$, $m < h$.

$$\tau K \imath_{m,n} A =$$

$$\tau < k_1, \ldots, k_m, \ldots, k_n, \ldots, k_j, \ldots, k_p > \imath_{mn} A =$$

$$\tau < k_1 \text{ ——————————— } a \text{ ————— } > \imath_{j,n}\, \imath_{m,n} A \equiv$$

$$\tau < k_1 \text{ ——————————————— } > \imath_{m,n}\, \imath_{j,n} A \equiv$$

$$\tau < k_1 \text{ —— } k_n \text{ ——————————— } > \imath_{j,n} A =$$

$$\tau < k_1 \text{ ——————————— } k_j \text{ ——— } k_p > A =$$

$$\tau K^1 A$$

h. $m,n < h$. Since j is the largest twin $h < j$. Hence $m \neq h$, $n \neq h$, $m \neq j$, $n \neq j$. So subcase b applies.

Case 3. K has no twins but it does have inversions, the largest of which is j. There are 14 subcases.

A. $m < n$

i) $k \neq m-1$, $k \neq m$, $k \neq n-1$, $k \neq n$

ii) $k = m-1$

iii) $k = m$ and $k < n-1$

iv) $k = m = n-1$

v) $m < k = n-1$

vi) $k = n,\ m = n-1$

vii) $k = n,\ m < n-1$

B. $n < m$

i) $k \neq n-1,\ k \neq n,\ k \neq m-1,\ k \neq m$

ii) $k = n-1$

iii) $k = n$ and $n < m-1$

iv) $k = n = m-1$

v) $k = m-1$ and $n < m-1$

vi) $k = m$ and $n < m-1$

vii) $k = m,\ n = m-1$

The proofs of these cases are all similar. We do A.iii and A.iv as examples.

A.iii) $\tau K \iota_{m,n} A =$

$$\tau\langle k_1, \ldots, k_m, \ldots, k_{n-1}, k_n, \ldots, k_p\rangle \iota_{m,n} A$$

$$\equiv \tau\langle k_1, \text{——}\ k_{m+1}, k_m\ \text{————————}\ k_p\rangle \rho_m \iota_{m,n} A$$

$$\equiv \tau\langle k_1\ \text{——————————————}\ k_p\rangle \iota_{m+1,n} \rho_m A$$

$$\equiv \tau\langle k_1\ \text{——————}\ k_n\ \text{——————}\ k_p\rangle \rho_m A \quad (\text{since } k_{m+1} < k_m)$$

$$\equiv \tau\langle k_1\ \text{——————}\ k_n, k_{m+1}\ \text{———}\ k_p\rangle A$$

$$= \tau K^1 A$$

A.iv) $\tau \mathbf{K} \ell_{m,n} A$

$= \tau\langle k_1, \ldots, k_{n-1}, k_n, k_{n+1}, \ldots, k_p\rangle \ell_{m,n} A$

$= \tau\langle k_1, \ldots, k_n, k_{n-1}, \text{———————} k_p\rangle p_{n-1} \ell_{n-1,n} A$

$\equiv \tau\langle k_1 \text{———————————————————} k_p\rangle \ell_{n,n-1} A$

$\equiv \tau\langle k_1 \text{————} k_n, k_n \text{———————} k_p\rangle A$

$= \tau \mathbf{K}^1 A$

Case 4. $\mathbf{K}$ has no twins or inversions but it does have absentees, the least of which is a.

a. $k_m \neq \min(k_1, \ldots, k_p)$

$\tau\langle k_1, \ldots, k_p\rangle \ell_{m,n} A =$

$\tau\langle a, k_1, \ldots, k_p\rangle [\ell_{m,n} A =$

$= \tau\langle a, k_1, \text{—————} k_n\rangle \ell_{m+1,n+1} [A$

$\equiv \tau\langle a, k_1 \text{————} k_n \text{————} k_n \text{————} k_p\rangle [A$

$\equiv \tau\langle k_1 \text{—————————————————————} \rangle \phi$

$= \tau\ ^1A$

b. $k_m = \min(k_1, \ldots, k_p)$. Since there are no inversions, $m = 1$.

$\tau \mathbf{K} \ell_{m,n} A =$

$\tau\langle k_1, \ldots, k_n, \ldots, k_p\rangle \ell_{1,n} A$

$= \tau\langle a, k_1 \text{————————} k_p\rangle [\ell_{1,n} A$

$\equiv \tau\langle a, \text{——————————} k_p\rangle \ell_{1,2} [\ell_{1,n} A$

$\equiv \tau\langle k_1, k_1 \text{————————} k_p\rangle [\ell_{1,n} A$

$$\equiv \tau\langle k_1, k_1 \text{ ———————— } k_p\rangle \iota_{2,n+1}[A$$

$$\equiv \tau\langle k,\ a \text{ ———————— } k_p\rangle \iota_{2,1}\iota_{2,n+1}[A$$

$$\equiv \tau\langle \qquad\qquad\qquad\qquad \rangle \iota_{2,n+1}[A$$

$$\equiv \tau\langle k_1, k_n \text{ ———————— } k_p\rangle [A$$

$$= \tau\langle k_n \text{ ———————— } k_p\rangle A$$

Lemma 4. $\mathcal{K} = \langle k_1, \ldots, k_p\rangle$

$\mathcal{K}^1 = \langle k_2, \ldots, k_p\rangle$

$\tau\mathcal{K}[A \equiv \tau\mathcal{K}^1 A$.

Proof. Induction on $\mathcal{K}$.

1. $\tau\mathcal{K} = \langle 1, \ldots, p\rangle$.

   $\tau\mathcal{K}[A =$

   $\tau\langle 1, \ldots, p\rangle[A =$

   $\tau\langle 2, \ldots, p\rangle A$.

2. $\mathcal{K}$ has twins, the largest of which is n and the largest twin of n in $\mathcal{K}$ is $\mathcal{M}$.

   a. $m > 1$

   $\tau\mathcal{K}[A =$

   $\tau\langle k_1, \ldots, k_m, \ldots, k_n, \ldots, k_p\rangle[A =$

   $\tau\langle k_1 \text{ ————— } a \text{ —— } k_p\rangle \iota_{n,m}[A \equiv$

   $\tau\langle k_1 \text{ ———————— } k_p\rangle[\iota_{n-1,m-1}A \equiv$

   $\tau\langle k_2 \text{ ———————— } k_p\rangle \iota_{n-1,m-1}A \equiv$

   $\tau\mathcal{K}^1A$

b. $m = 1$. Then k_1 is the only twin of k_n in K.

$\tau\langle k_1, \ldots, k_n, \ldots, k_p\rangle[A =$

$\tau\langle k_1, \text{———} \; a \; \text{———} \; k_p\rangle i_{n,1}[A \equiv$

$\tau\langle k_1 \text{——————————} k_p\rangle p i_{n,2} p[A \equiv$

$\tau\langle k_1 \text{——————————} k_p\rangle p i_{n,2} i_{2,1}[A \equiv$

$\tau\langle k_2\, k_2 \text{——} k_1 \text{——} k_p\rangle[A \equiv$

$\tau\langle k_2 \text{——————————} k_p\rangle A$ by induction hypothesis

(Since there is no twin in $\langle k_2 k_2 \text{—} k_1 \text{—} k_p\rangle$ as large as n).

3. K has no twins but it does have inversions, of which the smallest is j.

a. $j > 1$

$\tau\langle k_1, \ldots, k_j, \ldots, k_p\rangle[A =$

$\tau\langle k_1 \text{——} k_{j+1}, k_j - k_p\rangle p_j[A \equiv$

$\tau\langle k_1 \text{——————————} k_p\rangle[p_{j-1}A \equiv$

$\tau\langle k_2 \text{——————————} k_p\rangle p_{j-1}A \equiv$

$\tau K^1 A$

b. $j = 1$ and there is no absentee in K^1 which is less than k_1.

$\tau\langle k_1, \ldots, k_p\rangle[A =$

$\tau\langle k_2, \ldots, k_p\rangle A$.

c. $j = 1$ and there <u>is</u> such an absentee.

$\tau<k_1, k_2, \ldots, k_p>[A =$

$\tau<k_2, k_1$ —— $k_p>p[A \equiv$

$\tau<k_2, k_1$ —— $k_p>i_{2,1}[A \equiv$

$\tau<k_2, k_2$ —— $k_p>[A =$

$\tau<k_2, a$ —— $k_p>i_{2,1}[A \equiv$

$\tau<a, k_2$ —— $k_p>[A \equiv$

$\tau<k_2$ —— $k_p>A$.

4. K has no twins or inversions, but it does have absentees, the smallest of which is a.

$\tau K[A =$

$\tau<k_1, \ldots, k_n>[A =$

$\tau<a, k_1,$ — $k_n>[[A \equiv$

$\tau<k_1 a$ —— $k_n>$ $[[A \equiv$

$\tau<k_1 a$ —— $k_n>[[A \equiv$

τ $<a$ —— $k_n>[A \equiv$

τ $<k_2$ —— $k_n>A$.

<u>Lemma 5</u>. $p > n$. Then $\tau<k_1,\ldots,k_p>A^n \equiv \tau<k_1,\ldots,k_{p-1}>A^n$.

<u>Proof</u>. $\tau<k_1, \ldots, k_p>A^n \equiv$

$\tau<k_1, \ldots, k_p>P_p[A^n$ (by T29)

$$\tau\langle k_p, k_1, \ldots, k_{p-1}\rangle[A^n \equiv$$

$$\tau\langle k_1, \ldots, k_{p-1}\rangle A^n .$$

Corollary. $p > n \Rightarrow \tau\langle k_1,\ldots,k_p\rangle A^n \equiv \tau\langle k_1, \ldots, k_n\rangle A^n$.

Lemma 6. $k > k_1,\ldots,k_{p-1}$

$$K = \langle k_1,\ldots,k_{p-1},k\rangle \quad K^1 = \langle k_1,\ldots,k_{p-1}\rangle .$$

Then $\tau K A \equiv P_k P_p^{-1} \tau K^1 A$.

Proof. Since $k > k_1,\ldots,k_{p-1}$ the twins and inversions of K are the same as those of K^1. The lemma is proved by induction on K.

Lemma 7. Let $K = \langle k_1,\ldots,k_{p-1}\rangle$.

$$]P_p^{-1}\tau K P_p A \equiv \tau K]A .$$

Proof.

1. $]P_p^{-1} \tau\langle 1,\ldots,p-1\rangle P_p A \equiv$

$$P_p^{-1}PA \equiv]A \equiv \tau\langle 1,\ldots,p-1\rangle]A .$$

2. K has twins, the largest of which is n and the largest twin of n is m.

$$]P_p^{-1}\tau\langle k_1, \ldots, k_n, \ldots, k_{p-1}\rangle P_p A =$$

$$]P_p^{-1}\tau\langle k_1 \text{ ——— } a \text{ ——— } k_{p-1}\rangle \imath_{m,n} P_p A \equiv$$

$$]P_p^{-1}\tau\langle k_1 \text{ ————————— } k_{p-1}\rangle P_p \imath_{m+1,n+1} A \quad \text{(by T26)} \equiv$$

$$\tau{<}k_1 \;\text{———————}\; k_{p-1}{>}]\iota_{m+1,n+1}A$$

(by induction hypothesis) $\equiv$

$$\tau{<}k_1 \;\text{———————}\; k_{p-1}{>}\iota_{m,n}]A \;.$$

3. K has no twins, but it has inversions, the largest of which is j. This case is straightforward.

4. K has no twins or inversions but it has absentees, the least of which is a.

$$]\mathcal{P}_p^{-1}\tau{<}k_1,\ \ldots,\ k_{p-1}{>}\mathcal{P}_p\ A =$$

$$]\mathcal{P}_p^{-1}\tau{<}a,\ k_1,\ \ldots,\ k_{p-1}{>}[\mathcal{P}_p A \equiv$$

$$]\mathcal{P}_p^{-1}\tau{<}a \;\text{—————}\; k_{p-1}{>}\mathcal{P}_p \cdot p[A \quad \text{(by L27)} \equiv$$

$$\tau{<}a \;\text{—————}\; k_{p-1}{>}]\ [A \equiv$$

$$\tau{<}a \;\text{—————}\; k_{p-1}{>}[\,]A \quad \text{(by Ax24)} =$$

$$\tau{<}1 \;\text{—————}\; k_{p-1}{>}]A \;.$$

Lemma 8. P1-P12 are derivable from Ax1-Ax28 and R1-R3.

Proof. P1 is Ax1. P2 is true by definition. P3 follows from Lemma 5. P4 and P5 follow from Ax2 and Ax3 and the definition of τ. P6 and P7 are Ax20 and Ax21. P8 is Lemma 3. P9 follows from Lemma 2 and T28. P10 follows from Lemma 2. P11 is Lemma 4 and P12 follows from Ax32 and Lemma 4.

Lemma 9. PR3 is derivable.

Proof. Suppose $\vdash (A^m \cap \tau\langle k_o,\dots,k_n\rangle B^n) \to C^p$ where $k_o > m + n + p + k_1 + \dots + k_n$. Then $\vdash A \cap -C \to -\tau\langle k_o,\dots,k_n\rangle B$.

By Lemma 2 $\quad \vdash A \cap -C \to -\tau\langle k_1,\dots,k_n,k_o\rangle P_{n+1}B$.

By Lemma 6 $\quad \vdash A \cap -C \to -P_{k_o} P^{-1}_{n+1}\tau\langle k_1,\dots,k_n\rangle P_{n+1}B$.

By Ax2 $\quad \vdash A \cap -C \to P_{k_o} - P^{-1}_{n+1}\tau\langle k_1,\dots k_n\rangle P_{n+1}B$.

By T29 $\quad \vdash P_{k_o}[(A \cap -C) \to P_{k_o} - P^{-1}_{n+1}\tau\langle k_1,\dots,k_n\rangle P_{n+1}B$.

By Ax3 $\quad \vdash P_{k_o}(A \cap -C) \to P^{-1}_{n+1}\tau\langle k_1,\dots,k_n\rangle P_{n+1}B$.

By R3 and Ax3 $\vdash -]-[(A \cap -C) \to -]P^{-1}_{n+1}\tau\langle k_1,\dots,k_n\rangle P_{n+1}B$.

By T30 and Lemma 7 $\quad \vdash (A \cap -C) \to -\tau\langle k_1,\dots,k_n\rangle]B$.

By R1 $\quad \vdash A \cap \tau\langle k_1,\dots,k_n\rangle]B \to C$.

Since PR1 and PR2 follow from R1 and R2 we have now shown that the axiom system presented here is equivalent to the one presented in APL.