THE DOMINO RELATION:

FLATTENING A TWO-DIMENSIONAL LOGIC[1]

§1 Introduction

This paper concerns formal systems containing a connective $\Box$ with the truth definition:

$(x,y)\models\Box A$ iff $\forall z\ (y,z)\models A$.

Such systems can be regarded as modal logics with the familiar relational truth condition:

$w\models\Box A$ iff $\forall v\ (wRv \supset v\models A)$,

where the worlds w are ordered pairs and the relation R is defined by the condition $(x,y)R(x',y')$ iff $y=x'$. The relation so defined shall be called the domino relation. It is exactly the relation one tries to preserve in the game of dominos. The domino relation features in a number of recent discussions of "two-dimensional" modal logics. Section two of this paper surveys several of these applications. Section three describes conditions on a relation R that are necessary and sufficient for R to be regarded as the domino relation. This makes possible a "flattening" of two-dimensional logics based on the domino relation: the completeness problem for the two-dimensional logics is reduced to that for the one dimensional systems in which R satisfies the appropriate conditions. In section four such a

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1This paper is the result of a six year correspondence with Lloyd Humberstone of Monash University. It has benefitted in both substance and style from from many of Professor Humberstone's ideas. It has also benefitted from comments of Kit Fine.

logic is presented and axiomatized. The completeness proof employs two technical novelties--a technique of D.M. Gabbay for constructing irreflexive models is adapted to the construction of models satisfying other conditions, and a Henkin-style procedure is used to build a model as a whole rather than the worlds comprising the model. The final section outlines ways in which the system can be either refined or simplified.

§2 Absolute Necessity, Relative Necessity, Time and Quantifiers

An attractive idea that has recurred several times in the literature on modality[2] is that many (perhaps all) kinds of necessity can be construed as relativizations of a single absolute notion of necessity. Something is physically necessary if it is absolutely necessary that it follows from the laws of physics; legally necessary, if it is absolutely necessary that it follows from the legal code; and so on. But if modalities are given a relational semantics of the kind now commonly adopted, this idea is doomed to failure. Suppose, for example that $\bigcirc$ is some sort of "relative" necessity operator. Then according to the reductive scheme

$$\bigcirc A = \Box(L \supset A),$$

where $\Box$ is an absolute necessity operator and L is a sentence constant expressing an appropriate set of laws. Now suppose $\bigcirc$ satisfies the T schema, i.e., $\vdash \bigcirc A \supset A$. (This is surely

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[2]In [Humberstone] this idea is traced to various writings of A. R. Anderson, S. Kanger, and T. Smiley. In [Van Fraassen] it is attributed to "the nominalists and subsequently the empiricists".

appropriate for __some__ relative necessities.) Then in particular, $\vdash OL\supset L$, i.e., $\vdash \Box(L\supset L)\supset L$. But in any reasonable modal logic this implies $\vdash L$, and consequently that $\vdash OA\equiv\Box A$, i.e., that absolute and relative modality collapse. Semantically, the picture is the following. If $OA = \Box(L\supset A)$ then $w\models\Box A$ iff $v\models A$ for all L-worlds v such that wRv (where R is the accessibility relation for $\Box$). So we can interpret O by an accessibility relation R′ such that xR′y iff xRy and $y\models L$. But if R′ is reflexive (as would be expected if O satisfies the T schema) then $x\models L$ for every x, so R′=R.

These and similar difficulties with the modality-reduction project are noted by I.L. Humberstone[3] and B. Van Fraassen[4]. The two authors offer a similar diagnosis and remedy.[5] The difficulties arise because we are assuming that $\Box$ and L have "one-dimensional" interpretations. Allow "two-dimensional" connectives and constants and the project can easily be made to succeed. Humberstone's remedy employs the domino relation directly. Consider a relative necessity operator O interpreted by the accessibility relation R. Let $(x,y)\models\Box A$ iff $\forall z(y,z)\models A$, and let r be a sentential constant such that $(x,y)\models r$ iff Rxy. Then $(x,y)\models\Box(r\supset A)$ iff $\forall z(Ryz\supset(y,z)\models A)$. Taking A to be one-dimensional, and taking the truth of one-dimensional formulas to rest on

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3 [Humberstone]

4 [Van Fraassen]

5 Other remedies may also be possible. Wlodzimierz Rabinowicz, in correspondence, attributes to Lars Bergstrom the idea that, when $OA\supset A$ obtains, the reduction of relative necessity should be given by $OA = \Box(L\supset A)\&L$ rather than $OA = \Box(L\supset A)$.

second coordinates gives the desired reduction of ○ to □ plus constant.

Van Fraassen's remedy appears more drastic. He suggests that ○A should be rendered L→A where → is the binary two-dimensional connective with truth conditions

$$(x,y)\models A\to B \text{ iff } \forall z(\ (y,z)\models A \supset (x,z)\models B\ ).$$

$(x,y)\models A$ is to be understood as saying that the proposition which A expresses in the context of world x would be true in the circumstances of world y. Let us use the notation **[A]c** to indicate the proposition that A expresses in context c. Then [○A]c is true in circumstance w iff [A]c is true in all the circumstances in which [L]w is true. Now suppose L expresses, in any world, the physical laws of that world. Then ○A]c is true in circumstance w iff [A]c is true in all the circumstances in which the physical laws of w hold. ○A, in other words, always says that what A says is physically necessary.

Van Fraassen appears to provide a reduction of relative necessities to a special conditional, rather than a reduction to a special necessity operator. Van Fraassen's → cannot be defined in terms of Humberstone's □. But there is a connection between them. A formula is <u>valid</u>, for Van Fraassen, if it expresses in every world a proposition true in that world. Thus validity is truth at all pairs (w,w). This means that r→A is valid iff □(r⊃A) is valid. Similarly, consequence is preservation of truth at pairs (w,w). So the <u>logic</u> of r→A and □(r⊃A) is the same (as long as we do not permit nesting of □ or →.)

Humberstone's □, on Van Fraassen's understanding of the two-dimensional semantics, does express a kind of necessity. □A everywhere expresses a proposition that is true in exactly the circumstances in which A would express a proposition that is true in all circumstances. Identifying a proposition with the set of circumstances in which it is true, we can say that □A everywhere expresses the proposition that A is necessary. As Van Fraassen notes[6], □A is definable as ⊛⊞A, where:

(x,y)⊨⊛B iff (y,y)⊨B (⊛B expresses the proposition that B is true.)

(x,y)⊨⊞B iff ∀z (x,z)⊨B (⊞B expresses in every world the proposition that B is necessary in that world.)

But similar definitions in the other direction are also possible. For example: ⊞A = θ□A, where:

(x,y)⊨θB iff (x,x)⊨B (θB expresses in any world the proposition that B is true in that world.)

The domino relation appears in a quite different sort of role in Johan van Benthem's recent book on tense logic. Van Benthem notes that a relational frame (W,R) can be viewed as a directed graph with nodes in W and edges in R. But the same graph can be (at least partially) described by (X,S) where X is

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[6] I use □, ⊛, and ⊞ for Van Fraassen's ĝ, W, and Ɛ. The notation for the last two and for ⊛ follows [Segerberg], where such operators had previously been discussed with suggested readings "at the diagonal point on this longitude", "everywhere on this longitude" and "at the diagonal point on this latitude".

the set of edges and S holds between v1 and v2 if v1 terminates in a point from which v2 departs. In this case X can be taken to be a subset of WXW (viz., the set of pairs (w,v) such that there is an edge leading from w to v) and S to be the domino relation on X. The switch from (W,R) to (X,S) involves a kind of radical switch of metaphysical perspective. Connections between objects become objects themselves. The domino relation plays an important role in this shift.

The domino relation--or a generalization of it--appears in yet another guise in W.V. Quine's predicate functor logic. (See for example [] and [].) In predicate functor logic, the variables and functors of ordinary predicate logic are replaced by three or four special "functors" mapping predicates to predicates. The predicates and functors of predicate functor logic can be regarded as formulas and connectives of a many-dimensional modal logic. (See [] .) The functor] ("crop") that replaces the quantifiers has the following truth definition:

$$(x_1,\ldots,x_n)\models\,]A \text{ iff } \exists x_{n+1}\ (x_1,\ldots,x_{n+1})\models A.$$

], of course, corresponds to the <u>existential</u> quantifier. The system could just as easily have been formulated in terms of the <u>universal quantifier</u>:

$$(x_1,\ldots,x_n)\models\Box A \text{ iff } \forall x_{n+1}\ (x_1,\ldots,x_{n+1})\models A$$

If we restrict ourselves to a <u>dyadic</u> predicate logic in which the truth values of all formulas is determined by at most the last two indices7, this can be written:

$(x1,x2)\models\Box A$ iff $\forall x3\ (x2,x3)\models A$.

So the domino relation also underlies the interpretation of quantifier functors in dyadic predicate predicate functor logic.

§3 Domino structures and surrogate structures

For any non-empty set D, the domino structure on D is the pair (DXD, <) where (x,y)<(x′,y′) iff y=x′. A surrogate domino structure (or a surrogate structure for short) is a pair (W,R) were W is a nonempty set and R is a binary relation on W satisfying the following conditions. (We write uRvRw to indicate that uRv and vRw.)

SI (single intermediary) $\forall u,v\ \exists!w\ \ uRwRv$

IS (immediate successor) $\forall u\exists v(uRv \wedge \forall w(uRw\supset vRw))$

TS (transitive successor) $\forall u\exists v(uRv\ \&\ \forall w(vRw\supset uRw))$

Condition SI says that for any two members of D there is one and only one point that lies "between" them. This is equivalent to the conjunction of two-connectedness ($\forall u,v\exists w(uRwRv)$) and non-incestuality8 ($uRwRv\ \&\ uRw'Rv\ \supset\ w=w'$). Two-connectedness says

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7Such a system could be obtained, for example, by taking the two-place predicates, boolean functors, quantifier functors, and permutation functors from some version of Quine's system. Quine's "pad" functor could not be included since it generates predicates of more than two places.

8The term "incestual" is from [Chellas] p 82. **uRwRv** means **uRw and wRv.** Similar abbreviations are used subsequently.

that every pair of points is related by R2. Distinct points w and w′ are said to be "incestuous" if they have a common parent and a common child. Non-incestuality says there are no incestuous pairs. IS says that every point has an immediate successor, i.e., a successor that is related to all the points to which the original point is related. This condition is equivalent to the condition: $uRV \supset \exists w\, uRwRV$ (where uRV indicates that uRv for all v in V). When V contains only one element, this is just the familiar condition of density. Thus IS can be viewed as a generalization of density. To explain TS, let us say that u is <u>transitive through v</u> if $vRw \supset uRw$. In a transitive frame each point is transitive through all of its successors. TS says that in surrogate frames each point is transitive through some of its successors.

It is easy to check that domino structures are a species of surrogate frames. Let (DXD,<) be the domino structure on D. Given any two points (x,y) and (x′,y′) the unique intermediary is the point (y,x′). If (x,y) is related to all the points in Y then everything in Y must have y as a first coordinate. So (x,y)R(y,y)RY. Similarly, for any point (x,y), the point (y,y) is a successor of (x,y) through which (x,y) is transitive. So (DXD,<) meets the three defining conditions of surrogate frames.

More interesting is the fact that every surrogate structure is really a domino structure. To show this we first demonstrate that a number of properties follow from the three given in the definition of surrogate structures.

Lemma 1. Let ⟨W,R⟩ be a surrogate frame. Then R satisfies the following properties.

1. Separation of reflexives: $xRx \& yRy \& xRy \supset x=y$
2. Seriality: $\forall x \exists y\ xRy$
3. Reflexive successors: $\forall x \exists y\ (xRy \& yRy)$
4. Immediacy of reflexive successors: $xRy \& yRy \& xRz \supset yRz$
5. Uniqueness of reflexive successor: $xRyRy \& xRzRz \supset y=z$
6. Reflexive intermediaries: $xRy \& xRwRy \supset wRw$
7. Quasi-transitivity: $xRyRyRz \supset xRz$
8. Reflexive predecessors: $\forall x \exists y\ (yRx \& yRy)$
9. Uniqueness of reflexive predecessors: $yRyRx \& zRzRx \supset y=z$
10. Common successors: $xRu \& yRu \supset (xRv \equiv yRv)$

Proof

1. Suppose xRx, xRy and yRy. Then x and y are both intermediaries between x and y. So by SI, x=y.

2. Seriality is an immediate consequence of TS.

3. Let $Y=\{y:xRy\}$. By IS $\exists y\ xRyRY$. Since xRy, y must be a member of Y. Since yRY, yRy.

4. Suppose xRyRy and xRz. By IS there is a y' such that xRy'Ry and xRy'Rz. But now y' and y are both intermediaries between x and y, so by SI y'=y. Since y'Rz this means yRz.

5. Suppose xRyRy and xRzRz. By the immediacy of reflexive successors yRz. By the separation of reflexives y=z.

6. Suppose xRy and xRwRy. By reflexive successors there is a w' such that xRw'Rw'. By the immediacy of reflexive successors w'Ry. So w and w' are intermediaries of x and y and, by SI, w=w'. Substituting identicals we get wRw.

7. Suppose xRyRyRz. By TS there is a point y' through which x is transitive. By seriality, y' is related to some point z'. Since x is transitive through y' and xRy'Rz' we have xRz'. By reflexive intermediaries, then, y'Ry'. By uniqueness of reflexive successors y'=y. So x is transitive through y and xRz.

8. Consider an arbitrary point x. By SI there is a unique z such that xRzRx. By reflexive successor there is a y such that zRyRy. By the immediacy of reflexive successors yRx. So y is a reflexive predecessor of x.

9. Suppose yRyRx and zRzRx. By SI there is a unique w such that

xRwRy. In addition there is a unique v such that yRvRz. Now wRyRyRv, so by quasitransitivity wRv. Similarly vRzRzRx implies vRx. This means that v and y are both intermediaries of w and x. By SI, v=y. Substituting identicals we get yRz. By the separation of reflexives this means that y=z.

10. Suppose xRu and yRu. By SI and property 6 there are reflexive intermediaries w, between x and u, and w', between y and u. By property 9, w=w'. Now suppose xRv. w is a reflexive successor of x, so by the immediacy of reflexive successors wRv. By quasitransitivity, yRv. Similarly, yRv implies wRv, which implies xRv.

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Theorem 2. Every surrogate structure is isomorphic to a domino structure.

Proof Let M=(W,R) be a surrogate frame. We show that M is isomorphic to the frame determined by the domino structure on the set D of all w in W such that wRw. We use the notation bet(u,v) to denote the unique w such that uRwRv. Let f:DXD -- W be defined by f(u,v)=bet(u,v).

a. f is onto.

Let $w \in W$. We know w has a unique reflexive predecessor u and a unique reflexive successor v. Hence $u \in D$, $v \in V$ and f(u,v)=w.

b. f is one-one.

Suppose $f(x,y)=f(z,w)$. Then by uniqueness of reflexive predecessors $x=z$ and by uniqueness of reflexive successors $y=w$.

c. $(x,y) < (z,w)$ if and only if $f(x,y)\ R\ f(z,w)$.

i. Suppose $(x,y) < (z,w)$. Then $y=z$ and we have the situation pictured below.

By quasitransitivity, $f(x,y)Rf(z,w)$

ii. Suppose $f(x,y)\ R\ f(z,w)$.

Let $t=bet(y,z)$. By quasitransitivity $f(x,y)Rt$ and $tRf(z,w)$. By reflexive intermediaries tRt. By separation of reflexives $y=t=z$, and so $(x,y) < (z,w)$.

▩

Surrogate structures can be characterized in other ways as well. Consider the following conditions.

1. Branch points: $xRy \wedge xRz \supset \exists x'(xRx' \wedge x'Ry \wedge x'Rz)$

2. Right-identical successors: $\forall x\ \exists y\ (xRy \wedge \forall z(xRz \equiv yRz))$

Then each of the following sets of conditions characterizes surrogate structures:

a. Single intermediary + quasitransitivity + reflexive successors + branch points;

b. Right-identical successors + single intermediary.

Proof: First note that surrogate frames satisfy conditions 1 and 2 above. Branch points follows from generalized density. To prove right-identical successors, take an arbitrary point x and let y be its reflexive successor. By the immediacy of reflexive successors any successor of x is a successor of y, and by quasitransitivity any successor of y is a successor of x. Next show that the sets a and b each implies SI, IS and TS. Right-identical successors alone implies both immediate successors and transitive successors, so the conditions in b are clearly sufficient. From the fact that a contains SI, reflexive successors, and quasitransitivity, it follows that a implies TS. It remains only to check that a implies immediate successors. Take an arbitrary point x and let y be its reflexive successor. Suppose xRz. Then x "branches" to y and z, so there must be a "branch point" y′ that intervenes between x and its branches. If y′ were distinct from y, then y′ and y itself would both be intermediaries between x and y. So y′=y and therefore yRz. Since z was arbitrary, y is an immediate successor.

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The domino structures discussed above might be regarded as <u>full</u> domino structures; the structures that represent directed graphs are the substructures of these, which we may regard as partial domino structures. Thus a partial domino structure is a pair (W,<) where W is any non-empty set of ordered pairs and < is the domino relation on W. Let us define a partial surrogate structure to be a structure (W,R) satisfying the following conditions.

NI: $xRyRz \wedge xRy'Rz \supset y=y'$

CS: $xRy \wedge x'Ry \supset (xRz \supset x'Rz)$

NI is just the non-incestuality half of SI, mentioned above. CS ("common successor") says that if x and x' share one successor, they share all their successors. CS implies the analogous condition on predecessors. For suppose that y and y' have a common predecessor x, and suppose that y also has predecessor z. Then x and z share the successor y, and by CS, since x has y' as a successor, z must have y' for a successor as well. Note that part 10 of lemma 1 ensures that surrogate structures are partial surrogate structures.

It is routine to verify that all partial domino structures are partial surrogate structures. The other direction is established below.

Theorem 3. Every partial surrogate structure is isomorphic to a partial domino structure.

To prove theorem 2 we took the elements out of which the domino structure was constructed to be the reflexive members of the surrogate structure. In a partial surrogate structure these need not be present. We shall instead identify the basic elements with pairs of sets of members. In graphical terms the node x is to be identified with the pair (P,S) where P is the set of edges leading to x and S is the set of edges leading from x.

Suppose (W,R) is a partial surrogate structure. For any $w \in W$, the *successor-set* of w ($S(w)$) is the set of $v \in W$ for which wRv. The *predecessor-set* of w ($P(w)$) is the set of $v \in W$ for which vRw. An *interior node* is a pair (P,S) such that for some $p0 \in P$ and some $s0 \in S$, $P=P(s0)$ and $S=S(p0)$. By the common predecessor and common successor conditions, it follows that if (P,S) is an interior node then for any $p \in P$ and any $s \in S$, $S=S(p)$ and $P=P(s)$. An initial node is a pair $(\emptyset,\{w\})$ where w is a member of W with no predecessors and a final node is a pair $(\{w\},\emptyset)$ where w is a member of W with no successors. Let D be the set of all interior nodes, initial nodes and final nodes and let $U=\{((P,S),(P',S')) \in D \times D: S \cap P' \neq \emptyset\}$. Intuitively, U is the set of pairs of nodes x,y for which an edge leaves from x and leads to y. If $S \cap P'$ above is non-empty, it must contain exactly one member. (If there were two then (P,S) and (P',S') would both be intermediate nodes and there would be two intermediaries between the members of the non-empty sets P and S'.) We show that if $<$ is the domino relation on U, $(U,<)$ is isomorphic to (W,R). For all pairs $((P,S),(P',S'))$ in U, let $f((P,S),(P',S'))$ be the unique member of $S \cap P'$.

a. f is one-one.

Suppose $f((P,S),(P',S'))=f((Q,T),(Q',T'))=x$. Since $x\in S$, $P=P(x)$. Since $x\in T$, $Q=P(x)$. Hence $P=Q$. Similarly, since $x\in P'$ and $x\in Q'$, $S'=S(x)=T'$. If P (and hence Q) is empty then (P,S) and (Q,T) must be initial nodes and S and T must be singletons. But since they both contain x this ensures they are identical. If P (and hence Q) is non-empty then (P,S) and (Q,T) are interior nodes and S and T are the successor-sets of members of P and Q, so if P and Q are the same S and T are also the same. Similarly (P',S') and (Q',T') are either both final nodes, in which case they are both $(\{x\},\emptyset)$, or they are both interior nodes, in which case P' and Q' predecessor-sets of members of S' and T', and hence identical. Thus $((P,S),(P',S'))=((Q,T),(Q',T'))$.

b. f is onto.

Suppose $w\in W$. There are four cases.

(i) w has a predecessor p and a successor s. Let P,S,P',S' be $P(w),S(p),P(s),S(p)$. Then $((P,S),(P',S'))\in U$ and $f((P,S),(P',S'))=w$.

(ii) w has predecessor p, but no successor. Let P,S,P',S' be $P(w),S(p),\{w\},\emptyset$. Then $((P,S),(P',S'))\in U$ and $f((P,S),(P',S'))=w$.

(iii) w has successor s, but no predecessor. (similar to the above).

(iv) w has no predecessor and no successor. Let P,S,P',S' be $\emptyset,\{w\},\{w\},\emptyset$. Then $((P,S),(P',S'))\in U$ and $f((P,S),(P',S'))=w$.

c. $\alpha<\beta$ if and only if $f(\alpha)Rf(\beta)$.

Let $\alpha=((P,S),(P',S'))$, $\beta=((Q,T),(Q',T'))$, $f(\alpha)=x$ and $f(\beta)=y$. i.

Suppose $\alpha(\beta$. Then $(P',S')=(Q,T)$ and (P',S') is an interior node. Since $x \in P'$, $S'=S(x)$. But $y \in T$ and $T=S'$ so y must be a successor of x.

ii. Suppose xRy. $x \in P'$ and x has a successor so (P',S') must be an interior node. Hence $S'=S(x)$ and $P'=P(y)$. Similarly, $y \in T$ and y has a predecessor so (Q,T) must be an interior node. Hence $Q=P(y)$ and $T=S(x)$. Thus $(Q,T)=(P',S')$, i.e., $\alpha(\beta$.

The construction above, unlike the previous one, actually establishes a proper embedding of the surrogate structures in the "real" ones. Consider the two graphs below:

The "node structures" describing these are distinct, but the "edge structures" describing them are identical. The construction described above always results in a structure like the one on the right, rather than the one on the left. Thus the construction here actually shows that every partial surrogate structure is isomorphic to a partial domino structure with no branching from initial nodes and no branching to final nodes. (Note that it would not be possible for the construction always to select the structures like those on the left, for that would require that partial surrogate structures satisfy a sentence saying that there is at most one initial node and at most one final node.)

§4 The Logic of Absolute Necessity

In the last section it was shown that the full domino relations are just the relations satisfying SI, IS and TS. Consequently, the problem of axiomatizing Humberstone's $\Box$ is now reduced to the problem of finding modal axioms corresponding to these conditions.[9] The only condition that presents difficulties is the non-incestuality half of SI. There do not appear to be any simple modal axioms that preclude the existence of incestuous pairs in relational frames. The solution described here is an adaptation of an idea used by D.M. Gabbay[10] to construct irreflexive frames. There is no *axiom* that precludes the existence of reflexive points in models, but Gabbay shows that there are *rules* that do so. The axiomatization offered here will use similar rules for non-incestuality.

Let $\mathcal{L}$ be the usual language of modal logic, with p1,p2,... as sentence letters, $\vee$, $-$, and $\Box$ as primitive connectives, and $\wedge$, $\supset$, $\equiv$, and $\Diamond$ as defined connectives.

A (two-dimensional) *frame* for absolute necessity is a full domino structure (DXD, <). A (two-dimensional) *model* is a triple (DXD,<,V) where (DXD,<) is a frame and V is a valuation, i.e., a function that assigns a subset of DXD to each sentence letter. The model (DXD,<,V) is *based on* the frame (DXD,<). Truth at a pair, truth in a model and validity in a frame are defined in the

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[9] Similarly the logic of *partial* domino relations is the logic of relations satisfying NI and CP. Since every tree structure satisfies these conditions, the logic of partial domino relations is just K.

[10] [Gabbay]

usual way. (In particular, the clause for $\Box$ can be expressed: $(x,y)\models\Box A$ iff $\forall z\ (y,z)\models A$.) A <u>surrogate frame</u> for absolute necessity is a relational frame (W,R) satisfying SI, IS, and TS. A surrogate model is a relational model based on a surrogate frame. Truth at a point, truth in a surrogate model and validity in a surrogate frame are defined as usual. By the results of the previous section, the logic determined by the class of two-dimensional frames is the same as that determined by the class of surrogate frames.

$\Diamond n[A1,\ldots,An]$ abbreviates the formula $\Diamond(A1\wedge\Diamond(A2\wedge\ldots\Diamond(An\text{-}1\wedge\Diamond An)\ldots))$. $\Diamond n[A1,\ldots,An]$ is true at w1 if and only if there is a chain w1R...Rwn+1 such that A1,...,An are true at w2,...wn+1, respectively. The <u>logic of absolute necessity</u>, AN, is the logic determined by the axioms and rules below.

Axioms: A1. All substitution instances of tautologies.

A2. $\Box(A\supset B)\supset(\Box A\supset\Box B)$

A3. $\Box\Box A\supset A$

A4. $\Box\Box A\supset\Box\Box\Box A$

A5. $\Diamond\Box\Box A\supset\Box\Box A$

A6n. $\Box A0\wedge\Diamond A1\wedge\ldots\wedge\Diamond An\supset\Diamond(\Box A0\wedge\Diamond A1\wedge\ldots\wedge\Diamond An)$

Rules: MP. If $\vdash A$ and $\vdash A\supset B$ then $\vdash B$

Nec. If $\vdash A$ then $\vdash\Box A$

RNI. If $\vdash\Diamond n[A1,\ldots,An\text{-}1,\ An\wedge\Box(B\supset\Box q)\wedge\Box(-B\supset\Box\text{-}q)]\supset C$

then $\vdash\Diamond n[A1,\ldots,An-1,\ An\wedge\Diamond B\wedge\Diamond -B] \supset C$ (provided q does not occur in A1,...,An, B, or C.)

<u>Soundness</u>. Every theorem is valid in the class of all surrogate models.

Proof: Let M=(W,R,V) be a surrogate model. The truth of A1 and A2 in M is guaranteed by the fact that M is a relational model. The truth of A3, A4, and A5 is guaranteed by condition SI. (SI implies that every point is accessible in at most two steps from any other. This means that if □□A or ◇□□A is true anywhere, then A is true everywhere.) The truth, for all n, of A6n follows from the fact that R satisfies the right-identical successor property of the previous section. (Suppose that the antecedent of A6 is true at w. Let w′ be the right-identical successor of w. Since all the successors of w are successors of w′, w′ verifies ◇A1,...,◇An. Since all the successors of w′ are successors of w, w′ verifies □A0. Hence w verifies the consequent of A6.) Thus all the axioms are valid. Furthermore, it is clear that MP and Nec preserve validity.

All that remains to be shown is that RNI also preserves validity. Suppose some instance of the conclusion of that rule is false in the surrogate model (W,R,V). Then there are points x1,...,xn+1,y and y′ in W such that x1Rx2R...Rxn+1Ry, xn+1Ry′, A1,...,An are true at x2,...,xn+1, C is false at x1, B is true at y and B is false at y′. Let Y be the set of all points related to xn+1 at which B is true and let Y′ be the set of all points related to xn+1 at which B is false. Now since ⟨W,R,V⟩ satisfies non-incestuality, no point can be accessible from something in Y and something in Y′. That means we can let V′ assign all the successors of Y-points and none of the successors of Y′-points to the sentence letter q. If V′ agrees with V on all sentence letters other than q then ⟨W,R,V′⟩ will make A1,...,An true at x2,...,xn+1, C false at x1 and $\Box(B\supset\Box q) \wedge \Box(-B\supset\Box-q)$ true at xn+1. Thus ⟨W,R,V′⟩ will falsify the hypothesis of RNI.

■

Sufficiency Every finite set is satisfiable in some surrogate model.

Proof. Formulas of the form $\Diamond B\wedge\Diamond-B$ are special. If A is the special formula $\Diamond B\wedge\Diamond-B$ and q is a sentence letter, then A[q] is the formula $\Box(B\supset\Box q)\wedge\Box(-B\supset\Box-q)$. A set Γ has the NI property if whenever $A \in \Gamma$ for A special there is some sentence letter q such that $A[q] \in \Gamma$. The idea is to build a canonical model from maximal consistent sets with the NI property. The NI property will ensure that the model satisfies non-incestuality. Constructing the canonical model out of NI sets, however, requires the availability of "new" sentence letters. The problem is analogous to the problem of providing "witnesses" for existential formulas in completeness proofs of predicate logic. This accounts for the "simultaneous" construction of the worlds of the model mentioned in the introduction.

A Henkin system is a structure (W,R) where W=(W0,W1,...) is a countable sequence of (not necessarily distinct) sets of formulas and R is a binary relation on natural numbers that does not branch to the left, i.e., jRk and iRk implies i=j.

Suppose H=(W,R) is a Henkin system. A partial description of H from Wi is a member of the smallest set containing: 1)all conjunctions of formulas in Wi; 2)All formulas of the form A∧◇A1∧...∧◇An, where A is a conjunction of formulas in Wi and A1,...,An are partial descriptions of H from Wk1,...,Wkn respectively for k1,...,kn successors of i.

Now suppose a logic L closed under RNI is given. H is consistent (in L) if, for all i, every partial description of H from Wi is consistent. H is expandable if: 1) Wi is empty for all but finitely many i; 2) iRj implies Wi and Wj are not empty; 3) there are countably many sentence letters that do not occur in any formula in ∪{Wi:i<ω}. H is saturated if: 1)For all i, Wi is maximal consistent; 2)For all i, Wi has the NI property; 3)If ◇A∈Wi then, for some j, iRj and A∈Wj. H is an extension of the Henkin system (U,S) if Ui⊆Wi for every i and S⊆R.

Lemma 5. Every consistent expandable Henkin system has a saturated extension.

Proof. Suppose (U,S) is a consistent expandable Henkin system. Let (A1,n1),(A2,n2),... be an enumeration of all the pairs (A,n) such that A is a formula and n is a natural number. We construct the appropriate ⟨W,R⟩ from ⟨U,S⟩ in stages. At each stage i we consider adding Ai to Wni. More specifically, we proceed as follows:

i) ⟨W0,R0⟩=⟨U,S⟩

ii) Suppose ⟨Wn,Rn⟩ has been constructed and that ⟨A,m⟩ is the n+1'th pair in the enumeration. Let ⟨W',R'⟩ be the Henkin system obtained by adding A to Wnm (leaving Rn and Wnk, for k≠m, unchanged).

Case a) ⟨W',R'⟩ is not consistent. Then ⟨Wn+1,Rn+1⟩=⟨Wn,Rn⟩.

Case b) ⟨W',R'⟩ is consistent and A is not of the form ◇B or ◇B∧◇-B. Then ⟨Wn+1,Rn+1⟩=⟨W'R'⟩.

Case c) ⟨W',R'⟩ is consistent and A=◇B. Then ⟨Wn+1,Rn+1⟩ is obtained from ⟨Wn,Rn⟩ by adding ◇B to Wnm, adding B to Wnk for some k such that Wnk is empty, and adding the pair ⟨m,k⟩ to the relation Rn.

Case d) ⟨W',R'⟩ is consistent and A is special. Then ⟨Wn+1,Rn+1⟩ is obtained from ⟨Wn,Rn⟩ by adding both A and A[q] to Wnm where q is a sentence letter not occurring in any formula of ∪{Wni:i<ω}.

This construction clearly ensures that for every n ⟨Wn,Rn⟩ remains an expandable Henkin-system. We must check that consistency is preserved at every stage. Suppose ⟨Wn,Rn⟩ is consistent. If ⟨Wn+1,Rn+1⟩ is obtained under case a or case b above then it is consistent by definition. Suppose ⟨Wn+1,Rn+1⟩ is obtained under case c and that it is inconsistent. Then for some i there is an inconsistent partial description of ⟨Wn+1,Rn+1⟩ through i. We can think of each Henkin system as a collection of trees and available nodes. $\langle W_{n+1}, R_{n+1}\rangle$ is obtained by adding a single available node to the end of one branch of $\langle W', R'\rangle$. The only formula in the new end node is C where $\Diamond C$ was already in the old end node. So every partial description of $\langle W_{n+1}, R_{n+1}\rangle$ is also a partial description of $\langle W', R'\rangle$, and $\langle W', R'\rangle$ is not consistent. Finally suppose $\langle W_{n+1}, R_{n+1}\rangle$ is obtained under case d and that it is inconsistent. Let D be the inconsistent partial description, suppose D is a description of H from W_i, and let $k_1, \ldots, k_a$ be the sequence of nodes on the path from i to m (i.e., $i=k_1 R \ldots R k_a=m$). We can assume D is of the form $\Diamond a[C_1, \ldots, C_{a-1}, C_a \wedge A[q]]$, where each C_i ($1 \leq i < a$) is a conjunction of formulas in W_{k_i} and partial descriptions from W_j for $k_i R j$ and $j \neq k_{i+1}$, and C_a is a partial description from W_m. If D is inconsistent it follows by RNI that $\Diamond a[C_1, \ldots, C_{a-1}, C_a \wedge A]$, which is a partial description of $\langle W', R'\rangle$ is also inconsistent. Thus consistency is preserved in every case.

Now let $W_i = \cup\{W_{ni} : n < \omega\}$, $W = \langle W_1, W_2, \ldots\rangle$ and $R = \cup\{R_n : n < \omega\}$. We show that $\langle W, R\rangle$ is a saturated Henkin system. The only property that requires proof is that each W_i is maximal consistent.

Suppose W_m is not maximal, i.e., for some formula A, $A \notin W_m$ and $-A \notin W_m$. $\langle A, m\rangle$ and $\langle -A, m\rangle$ both occur in our enumeration of pairs, say at positions x and y. So it was inconsistent to add A to W_{xm} in $\langle W_x, R_x\rangle$ and it was inconsistent to add $-A$ to W_{ym} in $\langle W_y, R_y\rangle$. Since the construction is cumulative, at stages z greater than x and y it would be inconsistent to add either A or

-A to Wzm. Let D and D′ be the two inconsistent partial descriptions showing this fact. By adding conjuncts from D that are missing in D′ and vice versa we can insure that D and D′ are alike except that one contains an occurrence of A where the other contains an occurrence of -A. Furthermore the formula E that results from substituting a tautology for A at that position is a partial description of (Wz,Rz) and hence consistent. To see that this is not possible recall that every extension of K is complete for some class of tree models. Since E is AN-consistent there is a tree model that verifies E and the theorems of AN. E "describes" this model just as it describes (Wz,Rz). In fact those portions of the two structures described by E must be identical. The point in the tree model corresponding to Wzm must make either A or -A true. So it must be consistent to add either A or -A at the appropriate place in E. Thus (W,R) is maximal consistent and hence saturated.

If H=(W,R) is a Henkin system then the _model determined by_ H is the triple (N,R,V) where N is the set of natural numbers and $V(p)=\{n \in N: p \in Wn\}$. A routine induction establishes the following analog of the familiar result about canonical models.

Lemma 6. If M=(N,R,V) is the model determined by a saturated Henkin system then $(M,i) \models A$ iff $A \in Wi$.

From this lemma the following result follows.

Theorem 7. Every extension of K closed under RNI is weakly complete for a class of tree models with the property that for every world w the set of formulas true at w is an NI set.

Proof. Let L be an extension of K closed under RNI and let A be an L-consistent formula. Let $W=(\{A\},\varnothing,\varnothing,\ldots)$ and let $R=\varnothing$. Then (W,R) is an L-consistent Henkin system. By lemma 5, (W,R) can be extended to a saturated Henkin system (W',R'). By lemma 6 $(W'R')$ determines a model (N,R',V) that satisfies A. The model generated from (N,R',V) through 1 is the required tree model.

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<u>Theorem 7</u>. Every extension of K closed under RNI is weakly complete for a class of tree models with the property that for every world w the set of formulas true at w is an NI set.

Proof. Let L be an extension of K closed under RNI and let A be an L-consistent formula. Let $W=(\{A\},\emptyset,\emptyset,\ldots)$ and let $R=\emptyset$. Then (W,R) is an L-consistent Henkin system. By lemma 5, (W,R) can be extended to a saturated Henkin system (W′,R′). By lemma 6 (W′R′) determines a model (N,R′,V) that satisfies A. The model generated from (N,R′,V) through 1 is the required tree model.

The main object of study--the logic AN--is an extension of K closed under RNI. But surrogate models are not trees. It will be necessary to add some new points and merge some of the old ones to transform one kind of model into the other.

Let A be an AN-consistent formula. Let M=(W,R,V) be a tree model such that $(M,w0)\models A$ and for all w, $\{A: w\models A\}$ has the NI property. For each w in W let $|w|=\{A: w\models A\}$. Let $w1=\{B:\Box B\in|w|\}\cup\{\Box C:\Box C\in|w|\}\cup\{\Diamond D:\Diamond D\in|w|\}$.

<u>Claim:</u> w1 is consistent. Proof. If not, then $\vdash B1\wedge\ldots\wedge Bn\supset\neg(\Box C1\wedge\ldots\wedge\Box Cm\wedge\Diamond D1\wedge\ldots\wedge\Diamond Dk)$ where $B1,\ldots,Bn$, $\Box C1,\ldots,\Box Cm$, and $\Diamond D1,\ldots,\Diamond Dk$ are all in |w|. By A1,A2 and RNec, $\vdash\Box B1\wedge\ldots\wedge\Box Bn\supset\Box\neg(\Box C1\wedge\ldots\wedge\Box Cn\wedge\Diamond D1\wedge\ldots\wedge\Diamond Dk)$. Hence $\Box\neg(\Box C1\wedge\ldots\wedge\Box Cn\wedge\Diamond D1\wedge\ldots\wedge\Diamond Dk)\in|w|$. But since |w| is maximal consistent and $\Box Ci\in|w|$ for $1\leq i\leq n$, $\Box C1\wedge\ldots\wedge\Box Cn\wedge\Diamond D1\wedge\ldots\wedge\Diamond Dk\in|w|$. By A1 and A2, $\Box(C1\wedge\ldots\wedge Cn)\wedge\Diamond D1\wedge\ldots\wedge\Diamond Dk\in|w|$, and by A6 $\Diamond(\Box(C1\wedge\ldots\wedge Cn)\wedge\Diamond D1\wedge\ldots\wedge\Diamond Dk)\in|w|$. Further application of A1 and

A2 yields $\Diamond(\Box C1\wedge\ldots\wedge\Box Cn\wedge\Diamond D1\wedge\ldots\wedge\Diamond Dk)\in|w|$, which violates the consistency of $|w|$.

Now for all w in W let $\overline{w}$ be a maximal consistent extension of w1. A routine induction establishes that if A is <u>fully modal</u> in the sense that every sentence letter is in the scope of a $\Box$ or $\Diamond$, then $A\in\overline{w}$ if and only if $A\in|w|$. For any special formula A, A and A[q] are fully modal, and so each $\overline{w}$ will be NI. Note also that if $\Box A\in|w|$ then $\Box A\wedge A\in\overline{w}$. The $\overline{w}$'s are to be added to the model as immediate successors. More precisely:

$W' = W\cup\{\overline{w}: w\in W\}$.

$R' = R \cup \{(w,\overline{w}): w\in W\} \cup \{(\overline{w},v): w\in W \text{ and } wRv\} \cup \{(\overline{w},\overline{w}): w\in W\}$

$V'(p) = V(p) \cup \{\overline{w}: w\in W \text{ and } p\in|w|\}$

$M' = (W', R', V')$

Recall that for $w\in W$ $|w|=\{A:(M,w)\models A\}$. It is convenient to extend this notation to W' by stipulating that $|w|=w$ for $w \in W'-W$.

<u>Lemma 8</u>. $(M',w)\models A$ iff $A\in|w|$.

Proof. By formula induction. We do the case $A=\Box B$. Suppose first that $(M',w)\models\Box B$. There are two cases to consider.

a. $w\in W$. Then for all x in W', $wR'x$ implies $(M',x)\models B$. Since $R\subseteq R'$ this means that for all x in W wRx implies $(M',x)\models B$. By induction hypothesis wRx implies $B\in|x|$, i.e., wRx implies $(M,x)\models B$. Hence $(M,w)\models\Box B$, i.e., $\Box B\in|w|$.

b. $w\in W'-W$. Then $w=\overline{v}$ for some $v\in W$. $(M',\overline{v})\models\Box B$ implies $(M',x)\models B$ for all x such that $\overline{v}R'x$. But $\overline{v}$ is related to every y such that vR'y so vR'y implies $(M',y)\models A$. By induction hypothesis vR'y implies $A \in |y|$. Since $R \subseteq R'$, $y \in W$ and vRy implies $A \in |y|$, i.e., vRy implies $(M,y) \vDash A$. Hence $(M,v) \vDash A$, i.e., $\Box A \in |v|$. But since $|v|$ and $\bar{v}$ agree on fully modal formulas $\Box A \in |\bar{v}|$.

Now suppose $\Box B \in |w|$. Again there are two cases.

a. $w \in W$. Then $\Box B \in |w|$ implies $(M,w) \vDash \Box B$, which implies that, for all x in W, wRx only if $(M,x) \vDash B$, which implies that, for all x in W, wRx only if $B \in |x|$. We also know that $\Box B \in |w|$ implies $B \in \bar{w}$, and that if $w \in W$ the only R' successors of w are the R-successors and $\bar{w}$. Hence for all x in W', $wR'x$ only if $B \in |x|$. By induction hypothesis this means $wR'x$ only if $(M',x) \vDash B$, i.e., $(M',w) \vDash \Box B$.

b. $w \notin W$. Then $w=\bar{v}$ for some $v \in W$. $\Box B \in |w|$ implies $\Box B \in v$. The reasoning of case a above shows that vRx only if $(M',x) \vDash B$. But every R'-successor of <u>u</u> is also an R'-successor of v so this means $\bar{v}R'x$ only if $(M',x) \vDash B$, i.e., $(M',\bar{v}) \vDash \Box B$.

One consequence of Lemma 7 is that $(M',w0) \vDash A$. Another is that, for all fully modal formulas B, $(M',x) \vDash B$ iff $(M',\bar{x}) \vDash B$. The model must be modified further, however, before it can be proved to be a surrogate model.

Let $M''=(W'',R'',V'')$ be the model generated from M' through $w0$. For all v and w in W'', let $v \approx w$ if and only if u and v verify exactly the same formulas in M'. $\approx$ is an equivalence relation. Let $[w]$ be the equivalence class of w under $\approx$ and let $W+=\{[w]:w \in W''\}$. Let $V+(p)=\{[w]:w \in W''\}$. Let $[v]R+[w]$ if and only if, for all B, $(M',v) \vDash \Box B$ implies $(M',w) \vDash B$. Then $M+=(W+,R+,V+)$ is a filtration of M'' through the class of all formulas (in the sense of [Chellas], p101). It follows that $(M^+,[w])\models A$ iff $(M',w)\models A$. All that remains is to show that M^+ is a surrogate model.

1) M^+ is generated through $[w0]$

2) For all w in W $[\bar{w}]R^+[\bar{w}]$.

Proof. Since $\bar{w}R'\bar{w}$, $(M,\bar{w})\models\Box B$ implies $(M,w)\models A$.

3) R^+ satisfies IS.

Proof. If $x\in W$ then $[\bar{x}]$ is the immediate successor of $[x]$. For suppose $(M',x)\models\Box B$. Since $xR'\bar{x}$, $(M',\bar{x})\models B$, and so $[x]R^+[\bar{x}]$. If, in addition, $[x]R^+[y]$ then $(M',[\bar{x}])\models\Box B$ implies $(M',[x])\models\Box B$ (because $[x]$ and $[\bar{x}]$ agree on fully modal formulas), which implies $(M',[y])\models B$. So $[\bar{x}]$ is the immediate successor of $[x]$. If $x\notin W$ then $x=[\bar{v}]$ for some v in W. By fact 2 above, $[\bar{v}]R^+[\bar{v}]$, and hence x is its own immediate successor.

4) R^+ satisfies TS.

Proof. If $x\in W$ then $[\bar{x}]$ is a transitive successor of $[x]$. For it was shown in 3 above that $[\bar{x}]$ is a successor of $[x]$ and if $[\bar{x}]R^+[y]$ then $(M',x)\models\Box B$ implies $(M',\bar{x})\models\Box B$ (since x and $\bar{x}$ agree on modal formulas), which implies $(M^+,[\bar{x}])\models\Box B$, which implies $(M^+,[y])\models B$, which implies $(M',y)\models B$. Thus $[x]R^+[y]$, and so $[x]$ is transitive through $[\bar{x}]$. If $x\notin W$ then $x=[\bar{w}]$ for some $w\in W$. In this case x is itself a successor of x through which x is transitive.

5) R^+ is non-incestual. Suppose $[x]R^+[y]$ and $[x]R^+[y']$ for $[y]\neq[y']$. Since $[y]\neq[y']$ there must be an A such that $(M',y)\models A$ and $(M',y')\models -A$. In this case $(M',x)\models\Diamond A\wedge\Diamond -A$. Since $|x|$ is NI, $(M',x)\models\Box(A\supset\Box q)\wedge\Box(-A\supset\Box -q)$ for some sentence letter q. Hence

(M',y)⊨□q and (M',y')⊨□-q. If there were a z such that [y]R+[z] and [y']R+[z] then it would also be the case that (M',z)⊨q and (M',z)⊨-q, which is impossible.

6)R+ is two-connected. Note that by the definition of R+ [x](R+)n[y] if and only if (M',x)⊨□nB implies (M',y)⊨B and also [x](R+)n[y] if and only if (M',y)⊨B implies (M',x)⊨◇B. Let [x] and [y] be arbitrary members of W+ and suppose (M',x)⊨□□A. Since R+ is generated from [wO] we know that for some m and n [wO](R+)m[x] and [wO](R+)n[y]. Hence (M'wO)⊨◇m□□A. By m applications of A5, (M',wO)⊨□□A. By A4 (and also, if n<2, A3) this implies (M',wO)⊨□nA. Hence (M',y)⊨A and [x](R+)2[y].

§5 Analysis and synthesis

The system AN extends K by the addition of four axiom schemas and one rule schema. There is a sense in which it might have been more enlightening to have more axioms and a sense in which it might have been more enlightening to have fewer. A system with more axioms might provide a more finely grained analysis of the properties that enable a relation to be viewed as a domino relation; one with fewer axioms might provide a better means of grasping the properties as a synthetic whole.

A property may be said to be associated with a collection of axioms and rules if every normal extension of K containing the axioms and closed under the rules is complete for a class of models satisfying the property. It can be shown by standard arguments that the right-identical successor property is associated with the schema A6n. Similarly, it can be shown that TS is associated with the schema:

5) $\Box A \supset \Diamond \Box A$,

and that IS is associated with the schemas

D) $\Box A \supset \Diamond A$ and

ISA) $\Diamond A1 \wedge \ldots \wedge \Diamond An \supset \Diamond(\Diamond A1 \wedge \ldots \wedge An)$.[11]

The last three axioms are, of course, consequences of A6n. Since TS and IS, together with RNI, imply right-identical successor, it is natural to suppose that A6n might be replaced by these more "specific" axioms. I do not know whether this supposition is correct.

Progress in the other direction is easier to obtain. The formulas of AN fall into three categories: those whose truth at a pair (x,y) depends on both x and y, those whose truth at (x,y) depends only on y, and those whose truth at (x,y) depends on neither x nor y. It would be quite natural to take the language of AN to be explicitly sorted from the beginning. On this formulation $\mathcal{X}$ would contain three classes of sentence letters: the sort-2 letters p1,p2,..., the sort-1 letters q1,q2,... and the sort-3 letters r1,r2,... . (Humberstone's sentential constants characterizing different sorts of relative necessity would be sort-2 letters with fixed interpretation.) Sorted

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11A propery is said to <u>correspond</u> to a collection of axioms and rules if the frames satisfying the property are exactly the frames verifying the axioms and rules. It can also be shown that non-incestuality corresponds to RNI and that TS corrsponds to 5. IS and right-identical successor do not correspond to A6n and {D,ISA}, however, because there are frames lacking these properties that validate the axioms and rules.

formulas would be built up from the sentence letters by finite applications of the rules: for i=0,1,2, every sort-i sentence letter is a formula; if A is a sort i formula for $0<i\leq 3$ then $\Box A$ is a sort-(i-1) formula and if A is a sort-0 formula then $\Box A$ is a sort-0 formula; for i,j=0,1,2 if A is a sort-i formula then so is -A and if A and B are formulas of sort i and j then $A \wedge B$ are formulas of sort max(i,j). A valuation would assign truth-values, subsets of D and subsets of DXD to formulas of sort 0,1,and 2. The truth definition would be modified in the appropriate way, so that it would be easy to verify that for i=0,1,2 the truth value of a sort-i formula Ai at a pair depends only on the last i coordinates. Within this framework, A6 becomes an instance of the more general valid schema:

T◇1) $A1 \supset \Diamond A1$

(where A1 is a sort 1 formula). Similarly, A4 and A5 are instances of the valid schema:

Det0) $A0 \supset \Box A0$.

So the sorted version of the logic of absolute necessity is obtained by adding to K only two very simple axioms and the rule RNI.

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