On Connections Between Notions of Classical and Modern Modal Logic

(Some Basic Facts the Author Wishes He Had Known)

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July 30, 2018

The authors of [1] distinguish between the "classical" era of modal logic, stretching from 1959 to 1972, and the "modern" era, occuppying subsequent years. The classical era begins with the discovery of Kripke models and emphasizes use of such models to investigate completeness and decidability for logics of necessity, obligation, knowledge and time. The modern era begins with the identification of incomplete modal logics and is marked by deeper interest in the modal language and semantics for its own sake and its application in fields far from those that originally motivated it (especially computer science). Salient tools of the classical era include p-morphism (useful for completeness results) and filtration (useful for decidability results). Salient tools in the modern era (unsurprisingly, given their links to computer science) are bi-simulation and bi-simulation contraction. This note points out connections among these tools that are seldom, and in some cases never, made explicit. This should be useful for those who, like the author, were trained in classical times, but live and work in the modern world. One connection follows directly from the definitions: the p-morphisms from M to M' are just global bi-simulations between M and M' that are surjective functions. A familiar "modern" result, the Hennessy Milner theorem) reveals another connection: if its accessability relation is finitely branching then the bi-simulation contraction of M is just the finest filtration of M thourgh the set of all formulas. Modern investigation has also established that two models are globally bisimilar iff their bisimulation contractions are isomorphic. (This is noted in section 3.6 of [4] and it is gven focus and a nice exposition in [3].) Thus, for finitely branching models, bisimilarity just amounts to having the same finest filtration through the set of all formulas. Since every model is p-morphic to its bisimulation contraction, there is a connection between the classical and modern tools that holds even without the restriction: two models are globally bisimilar iff there is a model to which both are p-morphic. All of this is set out in more detail below. Definitions are provided for the notions under discussion and their important properties are identified, but, for the most part, the reader is referred elsewhere for proofs of results within the classical or modern traditions. We close by noting one further connection along similar lines: if accessibility relations are finitely branching, two models are bisimilar iff their coarsest filtrations are isomorphic.

1 Bisimulation and P-morphism

We work within a language $\mathcal{L}$ of sentential modal logic with sentence letters $p_1, p_2, \ldots$, and primitive connectives $\Box$, $\vee$ and $\neg$. $\Diamond$ and other connectives are defined as usual. We assume familiarity with the notion of a Kripke model $M = (W, R, V)$ suitable for $\mathcal{L}$ and the notion that formula A is true at world w in model M ($(M, w) \vDash A$). (See, for example, [2] whose notation we generally follow.) Henceforth, "model" will used to mean a Kripke model suitable for $\mathcal{L}$. If w is a world of model M, then we write $Th_M(w)$ for the *theory* of w in M, i.e., the set of formulas true at w in M.

Definition. *If $M = (W, R, V)$ and $M' = (W', R', V')$ are two models (not necessarily distinct), then $w{\in}W$ and $w'{\in}M'$ are are* equivalent *($w^M{\sim}w'^{M'}$) if they verify the same formulas, i.e., if $Th_M(w) = Th'_M(w')$. When confusion is unlikely we just say that w and w' are equivalent and write $w{\sim}w'$. M and M' are* equivalent *(written $M{\sim}M'$) if each world of M is equivalent to a world of M' and each world of M' is equivalent to a world of M.*

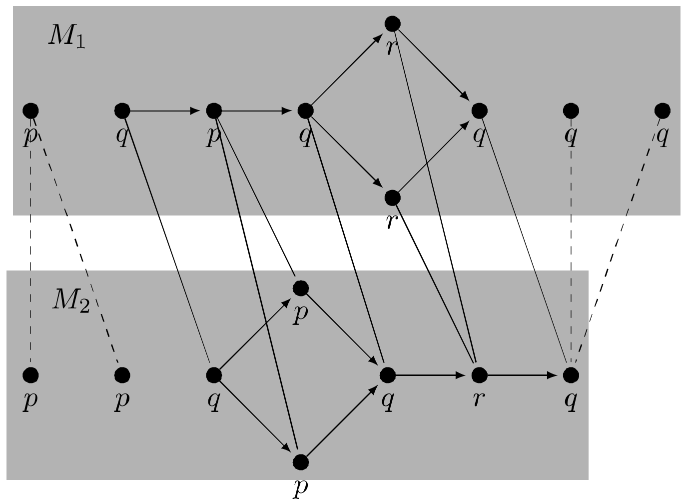
The definition implies that $\sim$, whether considered as a relation among worlds or models, is an equivalence relation.

Definition. *Suppose $M = (W, R, V)$ and $M' = (W', R', V')$ are models. Then a* p-morphism *from M to M' is function f from W onto W' such that: (i) $w{\in}V(p)$ iff $f(w){\in}V'(p)$, (ii) wRv implies $f(w)R'f(v)$, and (iii) $f(w)R'f(v)$ implies $\exists v'(f(v') = f(v)$ and $wRv')$.*

A p-morphism maps worlds to equivalent worlds and therefore models to equivalent models. (See, for example, [2] p98.) So, for example, knowing that the logic K is complete for the class of all models, to prove that it is complete for the class of tree-models it is sufficient to show that every model is a p-morphic image of some tree model. The notion of bisimulation gets at equivalence more directly.

Definition. *Suppose $M = (W, R, V)$ and $M' = (W', R', V')$ are models. Then a* bismulation *from M to M' is a relation Z between W and W' such that wZw' implies that: (i) $w \in V(p)$ iff $w' \in V'(p)$, (ii) wRv implies $\exists v' \in V'(vZv' \wedge w'R'v')$, and (iii) $w'R'v'$ implies $\exists v \in W(wRv \wedge vZv')$. Z is a* global bismulation *if, in addition, its domain is all of W and its counter-domain is all of W'. M is* bisimilar *to M' (written $M \leftrightarroweq M'$) if there is a bisimulation from M to M'.*

The diagram below illustrates these notions. The solid lines depict a bisimulation between models M and M'. When these are supplemented by the dashed lines the simulation depicted is global.



It follows immediately from the definition that bisimulation (unlike p-morphism) is symmetric and therefore that bisimilarity is as well. The definitions also immediately imply that p-morphism is a particular variety of bisimulation:

Theorem. *(i) Every p-morphism (considered as a relation) is a bisimulation. (ii) Every bi-simulation relation that is a surjective function is a p-morphism.*

2 Contraction and Filtration

A filtration of a model through a set of formulas Γ identifies worlds that agree on the truth values of formulas in Γ.

Definition. *Suppose $M = (W, R, V)$ is a model and Γ a set of formulas closed under subformulas. For all u and v in W let $u \sim_\Gamma v$ iff every formula* *of* Γ *gets the same truth value in* M *at* u *as it does at* v *and let* $[w]$ *be the equivalence class of* w *under* $\sim_\Gamma$. *A* filtration of M through Γ *is a model* $M^* = (W^*, R^*, V^*)$ *satisfying: (i)* $W^* = \{[w] : w{\in}W\}$*, (ii)* $V^*(p) = \{[w] : w{\in}V(p)\}$*, (iii)* uRv *implies* $[u]R^*[v]$ *and (iv)* $[u]R^*[v]$ *implies that, for all* $A{\in}\Gamma$*, if* $\Box A{\in}Th_M(u)$ *then* $A{\in}Th_M(v)$.

A routine induction establishes that if $M^* = (W^*, R^*, V^*)$ is a filtration of $M = (W, R, V)$ through Γ, then every formula in Γ gets the same truth value in$M$ at w as in M' at $[w']$. (See, for example, [2] pp101-102.) It follows that if M^* is a filtration of M through the set of all formulas then $M^*{\sim}M$. Let's call this the *filtration theorem*.

It is not obvious that models meeting the conditions defining filtration of $M = (W, R, V)$ through Γ exist. But suppose W^* and V^* are as in the definition and let $[u]R^*[v]$ iff for every $A \in \Gamma$, $\Box A{\in}Th_M(u)$ implies $A{\in}Th_M(v)$. Then conditions i, ii, and iv are satisfied by definition and condition iii follows from the truth definition of $\Box$. So $M^* = (W^*, R^*, V^*)$ is a filtration of M through Γ. Because R^* includes *every* pair that satisfies condition (iv), it is the largest relation that a filtration of M can have, and so M^* is the *coarsest* filtration of M. On the other hand, suppose R^* is defined so that $[u]R^*[v]$ iff $\exists u'{\in}[u]\exists v'{\in}[v](u'Rv')$. Then M satisfies conditions i, ii and iii. To check condition iv, suppose $[u]R^*[v]$ and $\Box A{\in}Th_M(u)$. By the definition of R^*, $\exists u'{\in}[u]\exists v'{\in}[v](u'Rv')$. Since $u'{\in}[u]$, $\Box A{\in}Th_M(u')$. By the truth conditions for $\Box$, $A{\in}Th_M(v')$. Since $v'{\in}[v], A{\in}Th_M(v)$, as required for condition (iv). So, in this case M^* is a filtration of M through Γ, in which R^* relates only the worlds required by condition (iii), and M^* is the *finest* filtration of M through Γ.

We turn now to the (modern) notion of *contraction*. First, we remind the reader of some salient properties of bisimulation, referring to the literature for proofs.

Theorem (Bisimulation)**.** *a) If* Z *is a bisimulation from* M *to* M' *and* wZw' *then* $Th_M(w) = Th'_M(w')$. *b) The set of bisimulations from to* M *to* M' *is closed under unions. Hence there is a largest bisimulation from* M *to* M'*, namely the union of all such bismulations. We use* $\rho^{M,M'}$ *for this bismulation. c)* $\rho^{M,M}$ *(henceforth written* ρ^M*) is an equivalence relation.*

Proof of a can be found, for example, in [1] p67 and proofs of b and c in [3] pp1-2. For our purposes a corollary of a is important.

Corollary. *If there is a global bisimulation between* M *and* M' *then* M *and* M' *are equivalent.*

Parts a-c of the bisimulation theorem make possible the following definition.

Definition. *The bismulation contraction of a model* $M = (W, R, V)$ *is the model* $M^\rho = (W^\rho, R^\rho, V^\rho)$ *where* ρ *is the largest bisimulation from* M *to* M*,* W^ρ *is the set of equivalence classes* $[w]^\rho$ *of members* w *of* W *under* ρ*,* $[u]^\rho R^\rho [v]^\rho$ *iff* $\exists v'(v\rho v'$ *and* $uRv')$ *and* $V^\rho(p) = \{[w]^\rho : w \in V(p)\}$

Part a ensures that V^ρ is well-defined. To check that R^ρ is well-defined suppose that $[u]R^\rho[v]$, $u\rho u'$ and $v\rho v'$. Because $[u]R^\rho[v]$, $\exists v''(v\rho v''$ and $uRv'')$. Because $u\rho u'$ and uRv'' (and ρ is a bisimulation), $\exists v'''(v''\rho v'''$ and $u'Rv''')$. Since ρ is an equivalence relation, this implies $\exists v'''(v'\rho v'''$ and $u'Rv''')$, i.e., that $[u']R^\rho[v']$.

We pause to record a connection between contraction and p-morphism.

Theorem (Contraction and P-morphism)**.** *Suppose* $M^\rho = (W^\rho, R^\rho, V^\rho)$ *is the bisimulation contraction of* $M = (W, R, V)$*. Then the function* $f : W \rightarrow W^\rho$ *is a p-morphism from* M *to* M^ρ

Proof. It follows from its definition that f is a surjective function and from the definition of V^ρ that f preserves truth-values of sentence letters. It remains only to check conditions (iii) and (iv) in the definition of p-morphism. For (iii), suppose uRv. Then there is a $v' \in [v]$ (namely v itself) such that uRv', and so $[u]R^\rho[v]$, as required. For (iv), suppose $[u]R^\rho[v]$. Then the definition of R^ρ immediately implies that uRv' for some $v' \in [v]$, as required. □

There is a well-known partial converse of part a of the bisimulation theorem, and a corollary.

Theorem (Hennessy Milner)**.** *If the accessibility relations of models* M *and* M' *are both finitely branching, and* $w \sim w'$ *then* wZw' *for some bisimulation* Z *from* M *to* M'*.*

Corollary. *If the accessibility relations of models* M *and* M' *are both finitely branching, then* $M \sim M'$ *iff* M *is globally bisimilar to* M'*.*

A proof of the Hennessy Milner theorem can be found, for example, in [1] p69. The right-to-left direction of the corollary follows from part a of the bisimulation theorem and the definitions of *global* bisimulation and $\sim$. For the other direction, suppose $M \sim M'$. Then by *Hennessy Miller*, for every world w of M and w' of M' there is a bisimulation Z from M to M' such that wZw'. By part b of the bisimulation theorem the union of these is a bisimulation showing that M is globally bisimilar to M' as required.

We can now restate and prove the previously mentioned connection between contractions and filtrations.

Theorem (Contraction and Filtration). *If its accessibility relation is finitely branching, the bisimulation contraction of a model M is just the finest filtration of M through the set of all formulas.*

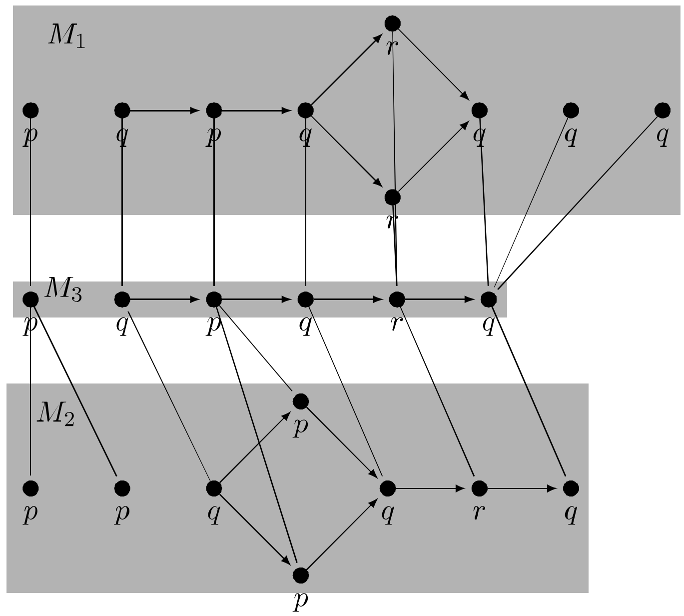
Proof. Recall that the bisimulation ρ^M is the union of all bisimulations from M to M. Hence wZw' for some bisimulation Z from M to M iff $w\rho^M w'$. By the Hennessy Milner theorem and its converse, it follows that if M is finitely branching then $w{\sim}w'$ iff $w\rho^M w'$. So the worlds of the bisimulation contraction of M are the same equivalence classes as those of the finest filtration of M through all formulas, and the valuation functions in the two models are identical. It remains only to check that $R^* = R^\rho$. But, by definition of R^ρ, $[u]R^\rho[v]$ implies $\exists v'{\in}[v](uRv')$, and, since $u{\in}[u]$, this implies $[u]R^*[v]$. Conversely, by definition of R^*, $[u]R^*[v]$ implies $\exists u'{\in}[u]\exists v'{\in}[v](u'Rv')$. By definition of R^ρ, this implies $\exists u'{\in}[u]([u']R^\rho[v])$, and, since $u\rho u'$, that $[u]R^*[v]$. $\square$

3 Bisimilarity, P-morphism and Filtration

We now restate modern the result about isomorphic contractions mentioned in the introduction, and illustrate it by extending the previous diagram.

Theorem (Isomorphic Contractions). *M is globally bisimilar to M' iff M^ρ is isomorphic to M'^ρ (where M^ρ and M'^ρ are the bisimulation contractions of M and M'.*

A proof can be found in [4] or [3]. To illustrate, we insert the common bisimulation contraction M_3 between the two globally bisimilar models M_1 and M_2 that were pictured above.



Putting *Isomorphic Contractions* together with *Contraction and Filtration* yields the following:

Corollary (Bisimulation and Filtration)**.** *If their accessibility relations are finitely branching, M is globally bisimilar to M' iff their finest filtrations through the set of all formulas are isomorphic.*

Putting *Isomorphic Contractions* together with *Contraction and P-morphism* yields the following:

Theorem. *$M \leftrightarrows M'$ iff there is some model N to which M and M' are p-morphic.*

Proof. The left-to-right direction follows immediately from the two theorems mentioned. For the right-to-left direction note that, since p-morphisms are bisimulations and bisimilarity is an equivalence relation, M and M' p-morphic to the same model implies that they are bisimlar. □

4 Bisimulation and Filtration Again

We noted above that, if accessibility relations are finitely branching then M and M' are globally bisimilar iff their finest filtrations through the set of all formulas are isomorphic. Here we observe that the same results holds when "finest filtrations" are replaced by "coarsest filtrations". The argument is by way of the (entirely classical) fact that equivalent models have isomorphic coarsest filtrations. Since, by *Hennesy Milner*, bisimilar models with finitely branching accessibility relations are equivalent, it follows that their coarsest filtrations are also isomorphic.

Theorem. *Suppose $M^* = (W^*, R^*, V^*)$ and $M'^* = (W'^*, R'^*, V'^*)$ are the coarsest filtrations through $\mathcal{L}$ of $M = (W, R, V)$ and $M' = (W', R', V')$. Then $M{\sim}M'$ iff M^* and M'^* are isomorpohic.*

Proof. For the left to right direction, suppose $M{\sim}M'$. Consider the set S of all world-theories, $Th_M(w)$, in M. Since M and M' are equivalent, S is also the set of world-theories in M'. By the filtration theorem, the world-theories of M and M^* are the same, as are those of M' and M'^* so S is the set of world-theories in all four models. In the filtration models, no two worlds have the same world-theories, so there is a bijection f from the worlds of M^* to those of M'^* that takes each $\boldsymbol{w}{\in}W^*$ to the unique $\boldsymbol{w'}{\in}W'^*$ with the same world-theory. Since f preserves world-theories, $[w]{\in}V^*(p)$ iff $f([w]){\in}V(p)$. It remains only to check that R is preserved under f. Suppose not $[u]R^*[v]$. Then by the definition of R^* there is a formula A such that $(M, u) \vDash \Box A$ but not $(M, v) \vDash A$. Since $u{\sim}[u]$ and $v{\sim}[v]$, $(M^*, [u]){\vDash}\Box A$ and not $(M^*, [v]) \vDash A$. Since f preserves world-theories, $(M'^*, f([u])){\vDash}\Box A$ and not $(M'^*, f([v])){\vDash}A$. By truth clause for $\Box$, not $f([u])R'^*f([v])$. A similar argument (with f^{-1} playing the role of f) establishes that not $f[u]R'^*f([v])$ implies not $[u]R^*[v]$.

For the converse, suppose M^* and M'^* are isomorphic. Then each world in one model is equivalent to its isomorphic image in the other so the models are equivalent. But, by the filtration theorem, $M^*{\sim}M$ and $M'^*{\sim}M'$. So $M{\sim}M'$. $\square$

Corollary (Bismulation and Filtration 2)**.** *a.If there is a global bisimulation from M to M' then the coarsest filtrations of M and M' are isomorphic. b. If the accessibility relations of models M and M' are both finitely branching, there is a global bisimulation from M to M' iff the coarsest filtrations of M and M' are isomorphic.*

References

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