Modal Logics that are Both Monotone and Antitone: Makinson's Extension Results and Affinities Between Logics

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Abstract

A notable early result of David Makinson establishes that every monotone modal logic can be extended to $L_{\mathbf{I}}$, $L_{\mathbf{V}}$ or $L_{\mathbf{F}}$, and every antitone logic, to $L_{\mathbf{N}}$, $L_{\mathbf{V}}$ or $L_{\mathbf{F}}$, where $L_{\mathbf{I}}$, $L_{\mathbf{N}}$, $L_{\mathbf{V}}$ and $L_{\mathbf{F}}$ are logics axiomatized, respectively, by the schemas $\Box\alpha \leftrightarrow \alpha$, $\Box\alpha \leftrightarrow \neg\alpha$, $\Box\alpha \leftrightarrow \top$ and $\Box\alpha \leftrightarrow \bot$. We investigate logics that are both monotone and antitone (hereafter amphitone). There are exactly three: $L_{\mathbf{V}}$, $L_{\mathbf{F}}$ and the minimum amphitone logic AM axiomatized by the schema $\Box\alpha \to \Box\beta$. These logics, along with $L_{\mathbf{I}}$, $L_{\mathbf{N}}$ and a wider class of "extensional" logics, bear close affinities to classical propositional logic. Characterizing those affinities reveals differences among several accounts of equivalence between logics. Some results about amphitone logics do not carry over when logics are construed as consequence or generalized ("multiple-conclusion") consequence relations on languages that may lack some or all of the non-modal connectives. We close by discussing these divergences and conditions under which our results do carry over.

1 Introduction

Marking the fifty years that have passed since the publication of Makinson [40], we provide some elaboration of one of its observations.[1] Fix a language with some truth-functionally complete set of (non-modal) connectives and one additional 1-ary connective $\Box$. Following Makinson, we mainly work with the notion of a modal logic as a set of formulas of this language that contains all tautologies and is closed under uniform substitution (of arbitrary formulas for sentence letters) and modus ponens. Minimally adapting the terminology of [40], we call a modal logic L (1) *congruential*, (2) *monotone* or (3) *antitone*, according as $\Box\alpha \to \Box\beta \in L$ whenever

[1]This paper of Makinson's has been the subject of much attention, a noteworthy recent example being Fritz [12], which sketches some general morals of [40], as well as focusing on a specific topic – Post completeness (understood as a property of a logic relative to some background class of logics) – that we shall mostly not engage with here. (Exception: Remark 2.4(*ii*).) For an appreciation of Makinson's logical achievements more generally, see the 'Outstanding Contributions to Logic' collection, Hansson [15].

(1) $\alpha \leftrightarrow \beta \in L$, (2) $\alpha \to \beta \in L$, or (3) $\beta \to \alpha \in L$, for all formulas α, β. When a formula α is a member of a logic L, we describe α as provable in L. (The three conditions on modal logics can all be formulated in axiomatic terms, the closure conditions just amounting to the admissibility of certain formula-to-formula rules.) A *consistent* modal logic is one in which not every formula is provable. As in [40], only congruential modal logics are under consideration here. Thus, in what follows, the word 'logic' will always, unless otherwise indicated, mean a congruential modal logic in a language with one 1-ary modal connective. The restriction to congruential logics allows us to by-pass complications over the choice of non-modal primitives noted in Makinson [41], and to link provability in modal logics to validity in modal algebras and, in Section 2, to validity on neighborhood frames. Here a *modal algebra* is simply a Boolean algebra (with the associated partial order denoted by $\leq$) expanded by an additional 1-place operation (written as $*$) for interpreting $\Box$. Makinson's concise presentation cannot be improved on, so we simply quote directly (from [40], p. 252):

> A modal algebra [with universe A] is said to be *monotonic* if for all $x, y \in A$, $x \leq y$ implies $*x \leq *y$, and is said to be *antitonic* if for all $x, y \in A$, $x \leq y$ implies $*y \leq *x$. Among the modal algebras there are clearly just four that can be obtained by adding a 1-ary operation to the two-element Boolean algebra: we shall call these the *unit* algebra ($*1 = 1$, $*0 = 1$), the *identity* algebra ($*1 = 1$, $*0 = 0$), the *complement* algebra ($*1 = 0$, $*0 = 1$), and the *zero* algebra ($*1 = 0$, $*1 = 0$). Each of these four algebras determines a corresponding set of formulae α such that for every homomorphism h from formulae into that algebra, $h(\alpha) = 1$.

Makinson notes that each of these four sets of formulas is a monotone or antitone modal logic, and shows (Theorem 2 of [40]) that any consistent monotone modal logic is a sublogic of the identity logic, the unit logic or the zero logic, and (Theorem 3 of [40]) that any consistent antitone (modal) logic is a sublogic of the complement logic, the unit logic or the zero logic. The four logics to which Makinson draws attention (henceforth 'Makinson logics') can be regarded as the logics obtained by interpreting $\Box$ according to one of the four 1-ary truth functions. More precisely, let **V**, **F**, **I** and **N** abbreviate the names Verum, Falsum, Identity and Negation, of the four 1-ary (bivalent) truth-functions. A *valuation* is a function assigning a truth-value to every formula of the language. A valuation is *Boolean* when this is done in accordance with the conventionally associated truth function for each connective for which there is such a convention: $v(\alpha \wedge \beta) = \mathrm{T}$ iff $v(\alpha) = v(\beta) = \mathrm{T}$, for all α, β, etc.; for present purposes this amounts to saying "for each non-modal connective". Each of the four Makinson logics is the set of formulas verified by all Boolean valuations that treat $\Box$ in accord with one of the four 1-ary truth functions, the complement logic, for example, being the set of formulas verified by Boolean valuations satisfying $v(\Box\alpha) = \mathrm{T}$ iff $v(\alpha) = \mathrm{F}$. For this reason, we refer to the four Makinson logics as $L_{\mathbf{V}}$, $L_{\mathbf{F}}$, $L_{\mathbf{I}}$ and $L_{\mathbf{N}}$. They are characterized, respectively, by the schemas $\Box\alpha \leftrightarrow \top$, $\Box\alpha \leftrightarrow \bot$, $\Box\alpha \leftrightarrow \alpha$, and $\Box\alpha \leftrightarrow \neg\alpha$.

The two normal Makinson logics, $L_{\mathbf{I}}$ and $L_{\mathbf{V}}$, are commonly known in the literature on normal modal logic as "the trivial logic" and "the Verum logic," respectively, or by some variation thereon.[2] One way to understand the Makinson results, mentioned in [18], is as saying that no monotone or antitone modal logic is contra-classical: in each such logic $\Box$ can be given a truth-functional interpretation relative to which the logic is sound (which is of course not to say: sound and complete). There is a striking contrast here with the larger range of congruential modal logics. Consider, for example, the 'demi-negation' logic discussed in Humberstone ([18] 451–456 and [25] 174–176). This is the smallest congruential logic containing all formulas of the form $\Box\Box\alpha \leftrightarrow \neg\alpha$. Adding any of $\Box\alpha \leftrightarrow \top$, $\Box\alpha \leftrightarrow \bot$, $\Box\alpha \leftrightarrow \alpha$, $\Box\alpha \leftrightarrow \neg\alpha$ to this logic results in inconsistency. (This is tantamount to the observation that composing any 1-ary truth-function with itself never results in the negation truth-function.) Hence demi-negation cannot be extended to any of the four logics Makinson singles out. A similar remark applies to logics containing $\neg\Box\bot \wedge \Box\Box\Box\bot$, as discussed in Humberstone ([24] and [25], pp. 174–176), and to some historically familiar logics like $\mathsf{S6}$. Indeed, as Theorem 5.2 of [12] implies, the maximal consistent extensions of such logics are not even congruential, and thus not among the logics to which attention is here restricted.

Makinson gives one extension result that does not involve a restriction to monotone or to antitone modal logics, namely Theorem 1 of [40], according to which any consistent congruential modal logic containing $\Box\top$ and $\neg\Box\bot$ is a sublogic of $L_{\mathbf{I}}$.[3] This is a generalization of the fact that any consistent extension of the normal modal logic KD is a sublogic of $L_{\mathbf{I}}$. Makinson's proof can easily be adjusted to prove that any congruential modal logic containing $\Box\bot$ and $\neg\Box\top$ is a sublogic of $L_{\mathbf{N}}$.

Here we continue to pursue the theme of Makinson's Theorems 2 and 3, with their focus on logics that are monotone or antitone, but consider a very restricted set of such logics, namely those which are both monotone *and* antitone. We refer to these as *amphitone*. In Section 2 we provide several characterizations of the class of amphitone logics and show how they fit within Makinson's picture. As it turns out, there are exactly three such logics. Each of these, and the members of a broader family to which they and the 'Makinson four' belong, have obvious affinities to classical propositional logic (henceforth 'CL'). In the case of the Makinson four, for example, the logical behavior of $\Box\alpha$ mirrors that of $\top$, $\bot$, α and $\neg\alpha$ in CL. In Section 3 we consider the nature of these affinities in more detail. In addition to shedding light on the modal logics of interest here, the discussion may contribute the understanding of notions like notational variance, and intertranslatability and 'logical synonymy', by providing simple examples where various analyses diverge. Section 4 investigates the extent to which our discussion of amphitony carries over when logics are construed as consequence relations or generalized (multiple-conclusion) consequence relations,[4] rather than as sets of sentences, and in which the underlying language

[2] A *normal* modal logic is a monotone modal logic containing $\Box\top$ and, for all formulas α, β, the formula $(\Box\alpha \wedge \Box\beta) \to \Box(\alpha \wedge \beta)$.

[3] Here $\top$ and $\bot$ can be taken either as primitive truth and falsity constants, or as abbreviating an arbitrary tautology or contradiction, respectively. In the sentence to which this note is appended we use the phrase *extension result* to cover what Makinson, in the title of [40], calls 'embedding theorems', since these results have nothing to do with embeddings in the sense of model theory or with the translational embeddings of Section 3 below.

[4] See [52] pp. 15, 29 and surrounding text, or [22] subsections 1.12 and 1.16 for relevant definitions

may lack some or all of the truth-functional connectives. We conclude in Section 5 with a brief summary and two open problems. The text of the paper is interspersed with numbered 'remarks'. The reader may skip these without missing the central points.

2 Amphitone Logics

In this section we provide one syntactic and two semantic characterizations of the amphitone logics. The former allows us to see how they fit within Makinson's extension results and the latter allow us give a complete inventory of them. Our syntactic characterization was noted already in [27] (pp. 39*f.*). Since the topic is central to our concerns here, we restate it as a theorem.

THEOREM 2.1 *A logic is amphitone iff it contains* $\Box\gamma \to \Box\delta$ *for all formulas* γ, δ.

Proof. Suppose S is amphitone. Because S is antitone and contains $\bot \to \gamma$, and $\delta \to \top$, it must contain both $\Box\gamma \to \Box\bot$ and $\Box\top \to \Box\delta$. Because it is monotone and contains $\bot \to \top$ it must contain $\Box\bot \to \Box\top$. Together, this implies it contains $\Box\gamma \to \Box\delta$. Conversely, if S *always* contains $\Box\gamma \to \Box\delta$ then it certainly does so when it contains $\gamma \to \delta$ or $\delta \to \gamma$. ■

Given Makinson's theorem, one might suspect that all amphitone logics are sublogics of $L_{\mathbf{V}}$ or $L_{\mathbf{F}}$. Being monotone, they are sublogics of $L_{\mathbf{I}}$ or $L_{\mathbf{F}}$. Being antitone, they are also sublogics of $L_{\mathbf{N}}$ or $L_{\mathbf{V}}$. But, although the $L_{\mathbf{I}}$ and $L_{\mathbf{N}}$ are themselves mutually inconsistent, nothing in [40] directly rules out the possibility that both conditions could be met by some consistent sublogic of $L_{\mathbf{I}}$ and $L_{\mathbf{N}}$ that is not also a sublogic of either $L_{\mathbf{V}}$ or $L_{\mathbf{F}}$. Nevertheless a slight addition to Makinson's result does allow us to confirm the suspicion mentioned.

THEOREM 2.2 *No antitone logic is a sublogic of* $L_{\mathbf{I}}$ *and no monotone logic is a sublogic of* $L_{\mathbf{N}}$.

Proof. For the first part, suppose L is antitone and a sublogic of $L_{\mathbf{I}}$. Since all logics in our sense contain $\bot \to \top$ as a member, and L is antitone, $\Box\top \to \Box\bot$ is a member of L, and, since L is a sublogic of $L_{\mathbf{I}}$, a member of $L_{\mathbf{I}}$ as well. But both $\Box\top \leftrightarrow \top$ and $\Box\bot \leftrightarrow \bot$ are members of $L_{\mathbf{I}}$. Together, this implies that $\top \to \bot$ is a member of $L_{\mathbf{I}}$, contradicting its consistency. The second part is proved similarly. ■

COROLLARY 2.3 *Every amphitone logic is a sublogic of* $L_{\mathbf{V}}$ *or* $L_{\mathbf{F}}$ *(or both).*

Proof. Since amphitone logics are monotone, Makinson's result tells us they are sublogics of $L_{\mathbf{I}}$, $L_{\mathbf{V}}$ or $L_{\mathbf{F}}$. Since they are antitone the theorem above implies they are not sublogics of $L_{\mathbf{I}}$. ■

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and further background.

Remarks 2.4 (*i*) Another proof of Corollary 2.3 uses a slightly refined version of an earlier (Kripke [32], Lemmon [37]) result that a modal logic is Halldén incomplete – i.e., proves some disjunction whose disjuncts share no sentence letters (alias propositional variables), without proving either disjunct – if and only if it is the intersection of two of its $\subseteq$-incomparable extensions. Let us say that α and β *witness* Halldén incompleteness in a logic L if α and β share no sentence letters and $\alpha \vee \beta \in L$ but neither α nor β is an element of L. Examining the proofs in [32] and [37] reveals that α and β witness Halldén incompleteness in L iff L is the intersection of $\subseteq$-incomparable extensions $L(\alpha)$ and $L(\beta)$, where these are the smallest extensions of L containing α and β, respectively. Now suppose L is both monotone and antitone. Consider the disjunction $\neg\Box p \vee \Box q$. Since L is amphitone, Theorem 2.1 ensures this is an element of L. If the left disjunct is provable, then (because L is consistent and $L_{\mathbf{F}}$ is Post complete) L is $L_{\mathbf{F}}$. Similarly, if the right disjunct is provable then L is $L_{\mathbf{V}}$. If neither disjunct is provable then $\neg\Box p$ and $\Box q$ witness Halldén incompleteness, so by the (refined) Kripke–Lemmon result, $L = L_{\mathbf{F}} \cap L_{\mathbf{V}}$. In all three cases, L has either $L_{\mathbf{F}}$ or $L_{\mathbf{V}}$ as an extension (possibly improper), as was to be shown.

(*ii*) The expression "Post complete" in (*i*) is intended to pick out the coatoms in the lattice of all modal logics. These are the "$\varnothing$-Post complete" logics of Fritz [12], where it is shown (Theorem 5.2) that the four Makinson logics comprise exactly the modal logics with this property, and we are reminded that there are uncountably many, as Fritz puts it, C-Post complete modal logics, meaning coatoms in the lattice of congruential modal logics, at least countably many of which are determined by classes of neighborhood frames. (See [12] Theorem 4.2.) In Humberstone [18], esp. note 10, a similar lattice-relativity is endorsed in connection with Post completeness, with the distinction drawn in the case of (e.g.) congruential modal logics between such a logic's being Post complete *qua* modal logic and its being Post complete *qua* congruential modal logic. ◀

Although a modal logic that is congruential, monotone or antitone may have extensions that are not (respectively) congruential, monotone or antitone (just as a normal modal logic can have non-normal extensions), a logic that is both monotone and antitone only has extensions that are themselves both monotone and antitone. This is an immediate consequence of Theorem 2.1. By contrast, the equivalence of being monotone and containing all instances of the schema $\Box(\alpha \wedge \beta) \to \Box\alpha$, for example, holds only across the range of congruential modal logics. And, as just noted, this class of modal logics is not itself closed under passing to extensions.[5]

Corollary 2.3 established that every consistent modal logic that is both monotone and antitone is a sublogic of $L_{\mathbf{V}}$ or $L_{\mathbf{F}}$ (or both). In this section we ask how many

[5]This has the following bearing on the Kripke–Lemmon characterization of Halldén incompleteness mentioned above: it is not true that a Halldén incomplete congruential, antitone, monotone or normal modal logic is the intersection of two of its $\subseteq$-incomparable congruential, antitone, monotone or normal (respectively) extensions. See p. 865 of Humberstone [22] for examples and discussion, concentrating on the normality case. The fact that being both monotone and antitone together is an extension-preserved property means that this potential failure of the Kripke–Lemmon characterization to 'transpose' to the current setting does not arise.

such logics there are. The answer, it turns out, is just three: $L_{\mathbf{V}}$, $L_{\mathbf{F}}$ and the minimal amphitone logic, which for brevity we will refer to as AM in what follows.[6] Given the framework within which we are operating, it follows by Theorem 2.1 that AM is the smallest congruential logic containing $\Box p \to \Box q$. Our proof that the amphitone logics comprise exactly these three (which appears as Corollary 2.7 below) uses a miniaturized version of the neighborhood semantics for modal logic. Recall that a neighborhood model is a structure $\langle W, N, V\rangle$ with W a non-empty set to each of whose elements the function N assigns a set of 'neighborhoods' – subsets of W – and V a map from the sentence letters of the language to subsets of W.[7] Truth of a formula α at $w \in W$ in such a model $\mathcal{M} = \langle W, N, V\rangle$ is notated as $\mathcal{M}, w \models \alpha$ and defined inductively as follows (where $\|\beta\|^{\mathcal{M}}$ is the set of elements of W at which β is true in $\mathcal{M}$):

- $\mathcal{M}, w \models p_i$ if and only if $w \in V(p_i)$.
- $\mathcal{M}, w \models \beta \wedge \gamma$ if and only if $\mathcal{M}, w \models \beta$ and $\mathcal{M}, w \models \gamma$, and similarly for negation and other non-modal connectives.
- $\mathcal{M}, w \models \Box\beta$ if and only if $\|\beta\|^{\mathcal{M}} \in N(w)$.

A formula is said to be *true in* $\mathcal{M}$ if it is true at every point in $\mathcal{M} = \langle W, N, V\rangle$, and *valid on* the frame $\langle W, N\rangle$ of such a model if it is true in every model on the frame. A modal logic is *determined by* a class of frames if its provable formulas are exactly those valid on each frame in the class. We say a logic is determined by the frame $\langle W, N\rangle$ when it is determined by $\{\langle W, N\rangle\}$.

We turn to the miniature version of this apparatus. From now on, we consider only frames $\langle W, N\rangle$ in which $|W| = 1$, because these one-element frames provide a convenient model-theoretic analog of the two-element modal algebras from the algebraic semantics in play in Makinson's discussion. To within isomorphism there are exactly four such frames, since if $W = \{w\}$ then $N(w)$ must be $\{W, \varnothing\}$, $\{\mathrm{W}\}$, $\{\varnothing\}$ or $\varnothing$. The logics determined by these frames are, respectively, $L_{\mathbf{V}}$, $L_{\mathbf{I}}$, $L_{\mathbf{N}}$, and $L_{\mathbf{F}}$. The first and last of these frames we shall call *constant* frames since in any model on either frame the truth value of $\Box\alpha$ at w is constant for varying α: either all $\Box$-formulas are true at w or all are false at w.[8] Since we are concerned here only

[6]This is not intended to be part of any systematic nomenclature – just a space-saving convenience for current purposes. The reader may regard it as abbreviating either <u>A</u>ntitone–<u>M</u>onotone or <u>AM</u>phitone.

[7]A valuable early discussion is provided by Hansson and Gärdenfors [14], using slightly different terminology – see the end of note 8. The more recent textbook treatment in Pacuit [43] may be consulted for more extended discussion, background, and variations.

[8]Calling the frame $\langle W, N\rangle$ 'constant' under precisely these circumstances may seem sloppy, since this is not a matter of N's being a constant function – as it must be whenever W is a singleton. To exhibit constancy at the frame level, one would pass from the conception of neighborhood frames as in our discussion, namely as pairs $\langle W, N\rangle$ with $N : W \longrightarrow \wp(\wp(W))$ to the equivalent structures $\langle W, f\rangle$ with $f : \wp(W) \longrightarrow \wp(W)$, as explained in Hansson and Gärdenfors [14], p. 157, in models on which f maps the set of points verifying a formula α to the set of points verifying $\Box\alpha$. The frames $\langle W, N\rangle$ we are calling constant are those for which in the corresponding frame $\langle W, f\rangle$, f is a constant function. ([14] has "U" in place of our "W" and calls neighborhood frames Scott-Montague frames.)

with one-element frames and models, we can drop the "w" and just write $\mathcal{M} \models \Box\alpha$ without ambiguity.

Lemma 2.5 *For any constant one-element model $\mathcal{M} = \langle W, N, V\rangle$ there is a substitution function $\mathfrak{s}$ such that, for any formula α, $\mathcal{M} \models \alpha$ iff $\mathfrak{s}(\alpha)$ is valid on the frame $\langle W, N\rangle$ of $\mathcal{M}$.*

Proof. For each sentence letter p_i, let $\mathfrak{s}(p_i) = \Box p_i$ if $\mathcal{M}$ assigns the same truth value to $\Box p_i$ and p_i, and let $\mathfrak{s}(p_i) = \neg\Box p_i$ otherwise. Then, for all formulas α, $\mathcal{M} \models \alpha$ iff $\mathcal{M} \models \mathfrak{s}(\alpha)$. (To see this note that, because $\mathfrak{s}(\Box\alpha) = \Box\mathfrak{s}(\alpha)$ and $\mathcal{M}$ is a constant model, it must be true for α of the form $\Box\beta$. The general claim then follows by a routine induction.) But $\mathfrak{s}(\alpha)$ is fully modalized, i.e., all its sentence letter occurrences lie within the scope of a $\Box$, so its truth in $\mathcal{M}$ does not depend on V (but only on N, and in particular on whether $N(w) = \{W, \varnothing\}$ or $N(w) = \varnothing$), and so $\mathcal{M} \models \alpha$ iff $\langle W, N\rangle \models \mathfrak{s}(\alpha)$, as was to be shown. ■

Theorem 2.6 *Every consistent monotone and antitone modal logic is determined by a non-empty class of constant one-element frames.*

Proof. Let L^+ be a modal logic extending AM, and Γ be some L^+-consistent set of formulas. Then Γ can be extended to a maximal L^+-consistent set of formulas $\Gamma^\star$, which for familiar reasons contains the consequent of any L^+-provable conditional whose antecedent it contains. Since $L^+ \supseteq \mathsf{AM}$, for all α, β whenever $\Box\alpha \in \Gamma^\star$, we also have $\Box\beta \in \Gamma^\star$. Thus $\Gamma^\star$ contains all $\Box$-formulas or none. This $\Gamma^\star$ will be the single element w of our one-element model $\mathcal{M} = \langle W, N, V\rangle$, with $N(w) = W$ if all $\Box$-formulas belong to $\Gamma^\star$ and $N(w) = \varnothing$ if no such formulas belong to $\Gamma^\star$. Finally $V(p_i) = w$ or $= \varnothing$ depending as $p_i \in \Gamma^\star$ or not. The usual induction on formula complexity establishes that $\mathcal{M}$ verifies exactly the formulas in $\Gamma^\star$, leaving us to check that all S^+-provable formulas are valid on the frame of $\mathcal{M}$. Suppose otherwise: i.e., that for some S^+-provable α there is a model $\mathcal{M}' = \langle W, N, V'\rangle$ on the same frame just constructed that falsifies α. In that case $\mathcal{M}' \models \neg\alpha$, so by Lemma 2.5 there is a substitution function $\mathfrak{s}$ with $\mathfrak{s}(\neg\alpha) = \neg\mathfrak{s}(\alpha)$ valid on $\langle W, N\rangle$. This implies that $\langle W, N, V\rangle \models \neg\mathfrak{s}(\alpha)$ and hence that $\langle W, N, V\rangle \not\models \mathfrak{s}(\alpha)$. Since α is provable in L^+, however, $\mathfrak{s}(\alpha)$ is as well, so this contradicts the fact that $\langle W, N, V\rangle$ (alias $\mathcal{M}$) is a model for L^+. ■

For brevity, let us denote the frame $\langle\{w\}, N(w)\rangle$ by subscripting the letter $\mathcal{F}$ with a description of $N(w)$. So the two constant such frames in this notation appear as $\mathcal{F}_{\{W,\varnothing\}}$ and $\mathcal{F}_\varnothing$.

Corollary 2.7 *There are exactly three consistent modal logics that are both monotone and antitone, namely* AM*,* $L_\mathbf{V}$*, and* $L_\mathbf{F}$*.*

Proof. The only nonempty classes of constant one-element frames are $\{\mathcal{F}_{\{W,\varnothing\}}, \mathcal{F}_\varnothing\}$, $\{\mathcal{F}_{\{W,\varnothing\}}\}$ and $\{\mathcal{F}_\varnothing\}$, which determine respectively the logics AM, $L_\mathbf{V}$ and $L_\mathbf{F}$. Since

$\{\mathcal{F}_{\{W,\varnothing\}}\}$ validates $\Box\bot$, $\{\mathcal{F}_\varnothing\}$ validates $\neg\Box\bot$, and $\{\mathcal{F}_{\{W,\varnothing\}}, \mathcal{F}_\varnothing\}$ validates neither of these formulas, these three logics are distinct. ■

We have used the neighborhood semantics here because it is in general currency. We could equally have run the argument with lighter structures, tailored to the job at hand. All that the one-element constant frames have really been doing for us is supplying, from two sources, V and N, information about what is happening to the non-modal and modal subformulas, respectively, of the formula to be evaluated. The latter are treated in a particularly uniform manner. So a more economical representation might simply consist in pairs $\langle \xi, \mathbf{a} \rangle$ where ξ is a truth value, T or F (or 1 or 0, if preferred), for the constant treatment of $\Box$-formulas, and $\mathbf{a}$ is a function assigning a truth value to each sentence letter. We could call such a pair a model, or more explicitly a *pair-model*, and, where $\mathcal{M}$ is the pair $\langle \xi, \mathbf{a} \rangle$, define truth for a formula α in $\mathcal{M}$ by induction on the complexity of α as follows:

- $\mathcal{M} \models p_i$ if and only if $\mathbf{a}(p_i) = \mathrm{T}$
- $\mathcal{M} \models \beta \wedge \gamma$ if and only if $\mathcal{M} \models \beta$ and $\mathcal{M} \models \gamma$, and likewise for negation and other non-modal connectives
- $\mathcal{M} \models \Box\beta$ if and only if $\xi = \mathrm{T}$.

Then we could re-run the above line of thought in terms of pair-models by isolating what was previously the frame of the model – the part that abstracts from the evaluation of the sentence letters – which would now be simply the ξ component of the pair $\langle \xi, \mathbf{a} \rangle$. For the proof of the analog of Corollary 2.7, note that there are three non-empty subsets of the set {T, F} of truth values-*qua*-'frames' and these subsets, {T, F}, {T} and {F}, play the roles of $\{\mathcal{F}_{\{W,\varnothing\}}, \mathcal{F}_\varnothing\}$, $\{\mathcal{F}_{\{W,\varnothing\}}\}$ and $\{\mathcal{F}_\varnothing\}$, in the proof of the Corollary. The $\mathbf{a}$ of the pair-model constructed in the analog of the proof of Theorem 2.6 would then be the assignment that maps to T just the sentence letters that are members of the $\Gamma^\star$ constructed in that proof. Lemma 2.5 would say that any formula true in a pair-model $\langle \xi, \mathbf{a} \rangle$ has a substitution instance valid on the model's 'frame,' ξ.

Since every amphitone logic contains AM, Corollary 2.7 provides a third proof for Corollary 2.3: all such logics have either $L_{\mathbf{V}}$ or $L_{\mathbf{F}}$ as extensions. Now we can see, however, that "all" encompasses only three.

Corollary 2.7 also facilitates another characterization of our minimal amphitone logic, AM.

Corollary 2.8 $\mathsf{AM} = L_{\mathbf{V}} \cap L_{\mathbf{F}}$.

Proof. Because the amphitone property is characterized by closure under rules, the amphitone logics are closed under intersection. By Corollary 2.7 $L_{\mathbf{V}}$ and $L_{\mathbf{F}}$ are amphitone, and so their intersection must be as well. But only the first of these logics contains $\Box\top$ and only the second contains $\neg\Box\top$, and so their intersection must be distinct from each. Hence that intersection must be the third of the amphitone logics enumerated in Corollary 2.7, namely AM. ■

Thus, when the logic AM is construed as a set of formulas, $\Box$ emerges as a 'hybrid' of (connectives for) the constant true and constant false truth functions, i.e., as a connective whose logical behavior is the common behaviour of the two.

Corollary 2.8, together with Makinson's two-element modal algebras and well-known facts about product matrices,[9] provides a further characterization of AM. It is the logic determined by the four-element matrix obtained by taking the direct product of Makinson's unit algebra and zero algebra in that (arbitrarily selected) order, with $\langle 1_U, 1_Z\rangle$ as designated element. (The subscripts here just register the modal algebras from which the top elements come). On this semantics, $\Box$ is interpreted as the constant function taking every element of the product algebra to $\langle 1_U, 0_Z\rangle$. The set of formulas valid in the matrix is the intersection of $L_{\mathbf{V}}$ and $L_{\mathbf{F}}$, which, as we have just seen, is AM.

This coincidence of $\Box$ being the hybrid of two truth-functional connectives and its being the logic being determined by the direct product of the corresponding algebras in Makinson fails when logics are construed as consequence relations.[10] Take $\vdash_{\mathsf{AM}}$ to be the consequence relation for which

$$\Gamma \vdash_{\mathsf{AM}} \beta \text{ iff } (\alpha_1 \wedge \ldots \wedge \alpha_n) \to \beta \in \mathsf{AM} \text{ for some } \alpha_1, \ldots, \alpha_n \text{ in } \Gamma.$$

Then β is an $\vdash_{\mathsf{AM}}$-consequence of the α_i in this sense just in case it is a consequence of them whether (in all its occurrences) $\Box$ is given a verum interpretation or given a falsum interpretation.[11] Using notation from Section 3 below, we put this by saying that $\vdash_{\mathsf{AM}} = \vdash_{\mathbf{V}} \cap \vdash_{\mathbf{F}}$. (Indeed in the more comprehensive nomenclature described there, $\vdash_{\mathsf{AM}}$ gets called $\vdash_{\mathbf{VF}}$.) But by contrast with the set-of-formulas logics, where the logic determined by the product matrix coincides with the intersection of the logics determined by the factor matrices (slogan form: "products are hybrids"), the same cannot be said for the corresponding logics *as consequence relations*: the consequence relation determined by the product matrix is not the intersection of the consequence relations determined by the factor matrices. For example, for this 'product' consequence relation $\vdash_{\otimes}$, say, we have $\Box\alpha \vdash_{\otimes} \beta$ for all α, β because $\Box\alpha$ never assumes the designated value $\langle 1_U, 1_Z\rangle$. But p is not a consequence of $\Box q$ in the unit algebra's consequence relation, and therefore $\Box q \nvdash_{\mathsf{AM}} p$.[12]

[9]In particular: the fact that the set of formulas valid in the direct product of two matrices is the intersection of the sets valid in the respective factor matrices. (See Corollary 2.12.2 in [22], for instance.)

[10]See note 4 for background.

[11]Note that $\vdash_{\mathsf{AM}}$ as defined here satisfies $\alpha \vdash \beta$ iff $\vdash \alpha \to \beta$, which means that we do not have to distinguish being monotone in the sense that $\Box\alpha \to \Box\beta$ is a consequence of the empty set whenever $\alpha \to \beta$ is, from being monotone in the sense that $\Box\beta$ is a consequence of $\Box\alpha$ whenever β is a consequence of α, and likewise, *mutatis mutandis* for *antitone*, *congruential*, and *amphitone*. This of course has nothing to do with monotonicity (aka *weakening*) as the purely structural condition on consequence relation that is relaxed in work on 'non-monotonic logic'.

[12]Here we use the "algebra" rather than "matrix" terminology for familiarity, identifying the four Makinson algebras with the matrices based on them with the top element (whether it is called 1 or T) as designated. The 'hybrid' terminology in this discussion is taken from subsection 3.24 of

Indeed, by using a characterization due originally to Łoś and Suszko [38],[13] it can easily be shown that the intersection consequence relation, which we may call $\vdash_{\mathbf{VF}}$ (see Section 3, or just read this as $\vdash_{\mathsf{AM}}$) has no characteristic matrix at all. Let us say that a consequence relation $\vdash$ has the *cancellation property* if, whenever $\Gamma, \bigcup\{\Delta_i : i \in I\} \vdash \alpha$ and, for each $i \in I$, Δ_i is $\vdash$-consistent (i.e., there is some α for which $\Delta_i \nvdash \alpha$), and the formulas of Δ_i have no sentence letters in common with those of Γ or those of Δ_j for $j \neq i$ or with α, then $\Gamma \vdash \alpha$. Then the result of Łoś and Suszko says that a consequence relation has a characteristic matrix iff it is closed under substitution and it has the cancellation property. (This holds even for $\vdash$ that, unlike those under consideration here, are not congruential.) As one instance of the cancellation property scheme we get – inserting the usually omitted braces and "$\varnothing$" for clarity – that if $\varnothing, \{\Box p\} \vdash \Box q$ and $\{\Box p\}$ is $\vdash$-consistent, then $\varnothing \vdash \Box q$. Since $\Box p \vdash_{\mathbf{VF}} \Box q$ and $\{\Box p\}$ is $\vdash_{\mathbf{VF}}$-consistent and not $\vdash_{\mathbf{VF}} \Box q$, $\vdash_{\mathbf{VF}}$ does not satisfy the cancellation condition, and so it has no characteristic matrix.

The situation is similar to that for Johansson's 'Minimal Logic' (henceforth ML), perhaps the most widely cited example of this phenomenon. The relevant instance of the cancellation property in this case is that if $\varnothing, \{p, \neg p\} \vdash \neg q$ and $\{p, \neg p\}$ is $\vdash$-consistent, then $\varnothing \vdash \neg q$. Since $\neg q$ is a consequence of $\{p, \neg p\}$ in ML, but (unnegated) q is not and since $\bot$ is not a theorem, it follows that ML has no characteristic matrix. The resemblance is increased when we think of $\neg\alpha$ as defined by $\alpha \to \bot$, with nullary $\bot$ since, although 1-ary, $\Box$ in AM is 'essentially nullary,' that is, the interpretation of $\Box\alpha$ is independent of α.

The observation that the interpretation of $\Box\alpha$ is independent of the interpretation of α suggests that in AM formulas of form $\Box\alpha$ might be regarded as varied representations of a single additional sentence letter in classical propositional logic (henceforth CL). Since the addition of a new sentence letter to the countable collection of those already present would not significantly change CL, one might suspect that AM is itself a kind of elaborately written version, or 'notational variant' even, of CL. Some caution is warranted here. A formula of the form $\Box\alpha$ can contain other formulas as subformulas, but a sentence letter cannot. Occurrences of $\Box\alpha$ that might be understood in AM as occurrences of a new sentence letter are those that are not themselves in the scope of another occurrence of $\Box$. Together with the sentence letters already occurring in α, these are α's *truth-functional constituents*, i.e., they are the formulas of which α is a truth-functional combination – 'atoms from the Boolean point of view', as it is put in Segerberg [50], p. 51. Other as-

[22], *q.v.* for further information on these matters, much of which originates with Rautenberg (who does not himself use that terminology): see [48] and subsequent papers, such as [49]. Providing syntactic characterizations of the various hybridized truth-functions is much easier for generalized consequence relations – as will be illustrated in Section 4 – than it is for consequence relations proper (i.e., without empty or multiple "right-hand sides"), which are Rautenberg's main concern.

[13] Initially, the result was not as widely known outside Poland as it deserved to be, and was rediscovered many years later in Shoesmith and Smiley [51]. Furthermore, Łoś and Suszko [38] misstated the result they proved. Both the duplication and the misstatement are chronicled in Wójcicki [55]. Our exposition below (including the 'cancellation' terminology) follows [51] more closely than [38]. Chapter 15 of [52] should be consulted for a more general cancellation condition, not exploiting some special features of the present case.

pects of our previous discussion also point to the close connection between AM and CL. We noted, for example, that in AM, $\Box$ can be regarded as a hybrid of the $\Box$ of $L_{\mathbf{V}}$ and the $\Box$ of $L_{\mathbf{F}}$, i.e. that AM contains exactly the formulas that are true on every Makinson valuation that assigns either T to all formulas of the form $\Box\alpha$ or F to all those formulas. (This appears more succinctly stated as the content of Corollary 2.8.) This implies that $\alpha \in \mathsf{AM}$ iff $\alpha(\top)$ and $\alpha(\bot)$ are both classical tautologies, where these are the results of substituting $\top$ and $\bot$ for all of α's modal truth-functional constituents. But this is reminiscent of a familiar fact. If $\beta(p_i)$ is a formula in the language of CL containing (at least) p_i as a sentence letter and $\beta(\top)$ and $\beta(\bot)$ are the results of substituting $\top$ and $\bot$ for p_i in $\beta(p_i)$, then $\beta(p_i) \in \mathsf{CL}$ iff $\beta(\top) \in \mathsf{CL}$ and $\beta(\bot) \in \mathsf{CL}$. So again, subformulas of the form $\Box\alpha$ in β are behaving like occurrences of a sentence letter p_i distinct from other sentence letters in β. The connection is suggested yet again by the pair-model semantics introduced after Corollary 2.7. A pair model's second coordinate, $\mathbf{a}$, interprets sentence letters, while its first coordinate, ξ, interprets $\Box$-formulas. $\mathbf{a}$ does its job by just assigning truth-values to sentence letters, whereas ξ does its by assigning a single truth value to *all* $\Box$-formulas. $\alpha \in \mathsf{AM}$ iff α comes out true under every assignment $\mathbf{a}$ of truth-values to sentence letters and every assignment ξ of a truth value to all the $\Box$-formulas. But this arrangement seems little different from one in which $\mathbf{a}$ alone assigns truth values to all the sentence letters and one additional expression standing in for all the $\Box\alpha$s.

The connection between modal logics and CL is of course even more obvious in the case of the Makinson logics. As we noted in the last paragraph of the introduction, $\Box\alpha$ behaves in $L_{\mathbf{V}}, L_{\mathbf{I}}, L_{\mathbf{N}}$ and $L_{\mathbf{F}}$, exactly as $\top, \alpha, \neg\alpha$ and $\bot$ behave in CL. Let us now consider in more detail the affinities that the modal logics considered here and their close relatives bear to CL.

3 Translations, Equivalence, Embedding

A number of proposals have been put forward to answer questions about when two logical systems represent the same logic, or synonymous logics or when they are notational variants. The literature on the subject is extensive and discussion is still active.[14] In this section we outline some central equivalence relations – or *affinities* as we have been calling them – among logics from this literature. We then ask which of these relations capture exactly how close the kinship is between CL and the logics under investigation here. Our discussion is summarized in two figures. Figure 1

[14] See, for example, Prawitz and Malmnäs [47], Kuhn [33] pp. 67–97, Segerberg [50] pp. 43–47, Epstein [8] pp. 375–399, Wójcicki [57] pp. 66–75, Pelletier and Urquhart [45], Kocurek [30], French [11], Wehmeier [54]. Since the present paper was submitted, another pertinent discussion has appeared (at least online), namely Meadows [42]; see also the references there supplied to work by J. Wigglesworth and by J. Woods. There is an independent, but closely related, literature on notions like "expressive power," "definability," and "uniform definability" that employs similar notions for cases when both logics are interpreted by the same class of models (and, in particular, when one is a sublogic of the other). For more on this, see Remark 3.2(i). There is also a sizeable literature concerning analogous notions for first order theories. Some examples appear at the end of note 30 below.

shows six notions of affinity and the known logical connections among them. Figure 2 displays the kinds of affinity that our logics bear to CL. The literature on the notions in focus in this section offers a wide range of terminology and a variety of definitions sometimes formulated in apparent ignorance of, and differing subtly from, one another. Numbered remarks in this section aim to situate our discussion within the literature.

One notion figuring prominently in the literature goes by the name of *translational equivalence*. For a precise definition, tailored for present purposes to apply to pairs L, L' of congruential modal logics, we make use of the following four conditions on such pairs, where s maps formulas of the language of L (over which the variable 'α' ranges) to formulas of the language of L' (over which 'γ' ranges), and t maps formulas of the language of L' to those of L. (We call any such maps translations from the one language to the other.)

1. $\alpha \in L$ only if $s(\alpha) \in L'$,
2. $\gamma \in L'$ only if $t(\gamma) \in L$,
3. $t(s(\alpha)) \leftrightarrow \alpha \in L$,
4. $s(t(\gamma)) \leftrightarrow \gamma \in L'$.

Recall that L and L' are each assumed to have a truth-functionally complete set of connectives, so the biconditional sign in the last two clauses is either primitive or definable in both. When conditions 1-4 are met, we call s and t *translations* from L into L' and L' into L. L is the *source* of translation s and L' is its *target*, and the labels are reversed for t.

DEFINITIONS 3.1 (*i*) s *embeds* L in L' (or 'L can be embedded in L' by s') if Condition 1 above is satisfied for all α. (Thus, re-lettering, Condition 2, taken as holding for all γ, says that t embeds L' in L.) We call translations s, t *embeddings* of L in L' and conversely when 1 and 2 are satisfied.

(*ii*) An embedding s is *faithful* if Condition 1 is satisfied when "only if" is strengthened to "if and only if."

(*iii*) A pair of embeddings s and t are *mutually inverse* when Conditions 3 and 4 are satisfied.

(*iv*) L and L' are *translationally equivalent* when there exist s, t satisfying Conditions 1–4 (for all formulas α, γ).

When s and t are mutually inverse, Conditions 1 and 2 imply that each embedding is faithful. Thus translational equivalence is bi-directional embeddability by mutually inverse translations or, equivalently, bi-directional faithful embeddability by such translations.

REMARKS 3.2 (*i*) Note that we take a translation to be any map from the formulas of one language into those of another. The term *translation* is often (as in Kuhn [33], Pelletier and Urquhart [45], Wójcicki [57]) reserved for maps satisfying some syntactic condition like compositionality, and sometimes (as in Kocurek [30], Epstein [8]) for those satisfying some logical condition like consequence-preservation. Our *translational equivalence* (or the analogous notion for consequence relations: see further Remark 3.4), which follows Kocurek [30], is *strong similarity* in Kuhn [33] and *syntactic equivalence* in Segerberg [50] and French [11]. Most attempts in the literature to characterize such notions as "same logic," "synonymous logic," "notational variant," involve translational equivalence under some restricted class of translations (like those discussed below); Kocurek ([30] pp. 295–296) usefully warns against presuming that talk of one logic being a notational variant of another should be susceptible to a single formal explication, since it amounts to saying that the differences between the one and the other are insignificant – something that could well depend on the particular topic under investigation. (Nevertheless, [30] makes clear that Kocurek is favorably disposed toward one candidate explication, which he calls 'schematic translational equivalence.') Two apparent exceptions to the claim that the relevant literature emphasizes relations of translational equivalence under an appropriately restricted class of translations are French [11] and Wehmeier [54]. French argues for an additional constraint of "external equivalence" – that the appropriate translational equivalence be preserved under the addition of new operators with the same properties to both logics. Wehmeier, who compares a wide range of equivalence relations, advocates understanding them as relations between *interpreted languages* rather than logics. That is, the approach assumes a shared set of models in which the common vocabulary of the languages compared is interpreted similarly. This is often a natural condition to impose on the discussion, and so the 'common models' approach is widespread: see for example Yang [59] and references there cited, as well as numerous papers by Jie Fan, for example [9]. But since this approach is potentially restrictive, we take a more neutral line here, not assuming any such common model-theoretic background. In any case, translational equivalence relations, appropriately restricted, do not disappear in these works. When French's vague phrase "same properties" is understood in syntactic terms and translations are *definitional* in the sense given in Definition 3.3(*iii*) below, then, as French indicates ([11], p. 329), translational equivalence already implies external equivalence. And Wehmeier's equivalence relations on interpreted languages induce corresponding translational equivalence relations on the logics they determine.

(*ii*) The reference to consequence-preservation in (*i*) is a reminder that most of the the work cited there treats translations as embedding one consequence relation in another, rather than one logic in the set-of-formulas sense in another, as in Definitions 3.1 above. Conditions 1–4 have obvious analogs for this more general case – 3, for example, becoming: $t(s(\alpha)) \dashv\vdash_L \alpha$. The reader preferring the consequence relation formulation is invited to understand the present discussion in those terms (as the discussion was in the closing paragraphs of Section 2 above). $\alpha_1, \ldots, \alpha_n \vdash_L \beta$ amounts to the L-provability of $(\alpha_1 \wedge \ldots \wedge \alpha_n) \to \beta$. (Thus when L is a normal modal logic, it is the local rather than the global consequence relation associated with L that is at issue.) One could also consider in place of consequence relations, generalized – alias multiple-conclusion – consequence relations, as the sources and targets of embeddings, though we shall not do so here. The question of whether to understand affinities between logics as conditions on sets of theorems or on equivalence relations is not always innocent, as will be noted in Remark 3.4 below. ◀

One should not presume that whether logics L and L' are translationally equivalent, or whether L is embeddable or faithfully embeddable in L', have anything do with their *logical strength* i.e., the extent of their provable formulas or sequents. Classical propositional logic, the modal logic $\mathsf{S5}$, and the trivial modal logic (alias $\mathsf{KT!}$ or Makinson's *identity* logic, $L_{\mathbf{I}}$, as we are calling it) are of strictly increasing logical strength, but the first and last are translationally equivalent. Many well-known translations, like those from classical to intuitionistic logic, faithfully embed stronger logics into weaker ones. Some find this phenomenon paradoxical. (For example, Béziau in [3].) The air of paradox dissipates when we realize that stronger logics typically make fewer logical distinctions among formulas, so that it is natural for the logical organization of the stronger logic to be a part of that of the weaker. (On the need for the word "typically" here, see [21].) Indeed, Pelletier and Urquhart ([45], p. 283) leave as an open problem the question of whether *any* weaker logic could be faithfully embedded in a stronger one. If one allows logics in different languages, this question has an easy affirmative answer. The classical propositional logic of the conditional, for example is faithfully embedded into the full classical propositional logic by the identity translation, and classical propositional logic is in turn faithfully embedded into any modal logic by the same translation. For that reason, where we say L is faithfully embeddable in L', Kuhn ([33], p. 75) says *L is a fragment of L'*. When the logics are both from the same language, the question is not so easy, but the answer is still affirmative, as shown in [20].

What gives translational equivalence some plausibility as a characterization of "the same logic" for a wide range of logics including those of interest here, is that one can show that, within this range, that two logics are translationally equivalent iff their Lindenbaum algebras are order-isomorphic.[15] Thus if we take logically equivalent formulas to be "saying the same thing" and we take a logic to be a way of organizing what can be said in terms of logical strength, then translationally equivalent logics provide the same organization. This same result, however, shows the limitation of that characterizaton. The order relation of the Lindenbaum algebra of a modal logic is just that characterized by its Boolean algebra reduct. Since there is (up to isomorphism) only one countable atomless Boolean algebra, virtually all logics in which the non-modal connectives get their usual (classical) interpretation will be translationally equivalent, unless the notion of translation is suitably restricted.[16]

Stricter conditions on translations than those introduced in Definitions 3.1 are obtained by attending to the degree to which they respect the internal structure of the formulas they translate. All but the last two of Definitions 3.3 below can be found in more or less the present terminology, in French's extensive taxonomy of

[15] See Kuhn [33] p. 69, Kocurek [30] p. 290.

[16] This was observed independently in Kuhn and Weatherson [35], Kocurek [30], and Paseau [44]. (Kocurek also discusses related work on this issue by E. Jeřábek.) It holds, for example, for system where, like all those considered here, every formula has countably many logical equivalents. As Kocurek observes, it can fail when the set logical equivalents for some formula in L and the set of logical equivalents of some formula in L' differ in cardinality.

translations.[17] Defs. 3.3(v) and 3.3(vi), adapted from the notion of being *compositional* in Kocurek [30] (meaning what we are calling 'broadly compositional') are included here for ease of comparison with Defs. 3.3(ii) – (iv), though they play an active role in our discussion only later.

DEFINITIONS 3.3 (i) A translation t from one language to another is *variable-fixed* if for each sentence letter p_i, $t(p_i) = p_i$;

(ii) t is *narrowly compositional* if for each n-ary primitive connective # in the source language there is a formula $\alpha(p_1, \ldots, p_n)$ in the target language containing occurrences of no sentence letters other than $p_1, \ldots, p_n$, such that when $\alpha_1, \ldots, \alpha_n$ map to $t(\alpha_1), \ldots, t(\alpha_n)$ then $\#(\alpha_1, \ldots, \alpha_n)$ maps to $\alpha(t(\alpha_1), \ldots, t(\alpha_n))$;

(iii) A translation t is *definitional* if t is both variable-fixed and narrowly compositional.

(iv) Logics that are translationally equivalent via definitional translations, we call *definitionally equivalent.*

(v) t is *broadly compositional* if for every primitive n-ary connective # of the source language there is an n-ary template $\beta(q_1, \ldots, q_n)$, i.e., a formula in the language obtained by adding to the target language n new sentence letters $q_1, \ldots, q_n$ to those already in that language, such that, whenever $\alpha_1, \ldots, \alpha_n$ are mapped to $t(\alpha_1), \ldots, t(\alpha_n)$, $\#(\alpha_1, \ldots, \alpha_n)$ is mapped to $\beta(t(\alpha_1), \ldots, t(\alpha_n))$.

(vi) Logics that are translationally equivalent by broadly compositional translations, we call *compositionally equivalent.*

For example, for t broadly compositional we may have $t(p_i) = p_{2i}$ and $t(\Box\alpha) = \neg(p_0 \vee t(\alpha))$, for t narrowly compositional we may have $t(p_i) = p_{2i}$ and $t(\Box\alpha) = \neg t(\alpha)$ and for t definitional we may have $t(p_i) = p_i$ and $t(\Box\alpha) = \neg t(\alpha)$. To distinguish the templates of Defs. 3.3(v) from those of note 17, we shall call the current versions, in which only the new sentence letters are exhibited, *templates*$^+$. 'Old' sentence letters may still occur in the formulas represented by such templates$^+$, but they are not substituted for in the course of applying the translation.[18] Notice that definitional translation, narrowly compositional translation, compositional translation and translation are increasing classes, and so definitional equivalence, compositional equivalence and translational equivalence are increasingly general equivalence relations.

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[17]See [10] Chapter II. The definitional translations (of Def. 3.3(iii)) are labeled *simple translations* in Kuhn [33]. The analog of the (narrow) compositionality condition of Def. 3.3(ii) in French [10] requires that what we may call the *template* formula $\alpha(p_1, \ldots, p_n)$ contain occurrences of all the displayed sentence letters. This difference is not significant for the logics we consider here, but our formulation allows us count as definitional a translation s such that $s(\Box\alpha) = \bot$, rather than requiring us to rewrite this as, say, $s(\Box\alpha) = s(\alpha) \wedge \neg s(\alpha)$. The reader should beware that both labels are sometimes used to pick out slightly different classes of translations by other authors. By taking a more concrete view, according to which the formulas of the object language have to be taken sequences of symbols (rather than elements of the absolutely free algebra generated by the sentence letters, with the primitive connectives as fundamental operations), Wehmeier [54] is able to consider substantially tighter restrictions.

[18]The present templates$^+$ are the *contexts* of Humberstone [23], p. 50*f.*, and Yang [59], Def. 3.3, the latter supplying additional references.

Remark 3.4 It is well known that these stronger equivalence relations, applied (as here) to logics as sets of formulas, may diverge from the analogous relations among logics as consequence relations (as described in Remark 3.2. For example, while there is a definitional translation noted by Gödel embedding CL faithfully in intuitionistic propositional logic (henceforth IL), there is no such translation faithfully embedding $\vdash_{\mathsf{CL}}$ in $\vdash_{\mathsf{IL}}$. This was shown in Tokarz and Wójcicki [53] (p. 123*f.*; the authors' own discussion is couched in terms of consequence operations rather than consequence relations). In fact, the latter claim remains correct even on deletion of the word 'faithfully' (Humberstone [18], p. 466*ff.*). There is of course a definitional translation – the identity translation – embedding $\vdash_{\mathsf{IL}}$ in $\vdash_{\mathsf{CL}}$, but there is no such translation effecting an embedding faithfully in this direction either. (See Theorem 3.1 in Humberstone [19].) French's remark ([11], p. 329) that "Wójcicki proves that there can be no faithful definitional embedding of intuitionistic logic ($\vdash_{\mathsf{IL}}$) into classical logic ($\vdash_{\mathsf{CL}}$)" is slightly misleading. It is actually the converse direction of embedding that Theorem 1.8.9 of Wójcicki [57] (recapitulating the reasoning of [53]) bears on. Wójcicki was well aware of both directions, however, as his reference to Wojtylak [58] on p. 75 of [57] shows. Theorem 21(i) of [58] says, in the present terminology, that if there is a faithful definitional translation embedding an intermediate consequence relation $\vdash$ (meaning that $\vdash_{\mathsf{IL}} \subseteq \vdash \subseteq \vdash_{\mathsf{CL}}$) in an intermediate consequence relation $\vdash'$, then $\vdash = \vdash'$. For the present application, take $\vdash, \vdash'$ as $\vdash_{\mathsf{CL}}, \vdash_{\mathsf{IL}}$. (The relevant result was also cited in Paragraph 67.5 of Wójcicki [56], together with the mysterious reference "Wojtylak [1979b]," which is absent from [56]'s bibliography and evidently intended to be a reference to [58].) The observation that no definitional translation faithfully embeds $\vdash_{\mathsf{IL}}$ in $\vdash_{\mathsf{CL}}$ is strengthened in Theorem 24 of Kocurek [30] to the result that there is not even a broadly compositional translation that does this. (In fact Kocurek's result is stronger still, formulated in terms of what he calls schematic translations – not defined here, though alluded to Remark 3.2(i) – of which broadly compositional translations are a special case.) ◀

Since our interest here concerns the affinity of various modal logics to CL we impose the additional constraint that all translations are what we might call $\Box$-*translations*, i.e., that they respect the non-modal connectives, in the sense that, for example, $t(\alpha \wedge \beta) = t(\alpha) \wedge t(\beta)$.[19] In surveying the logics bearing some affinity to CL, for example, we would not want to count both AM and a variant in which '$\wedge$' indicated disjunction and '$\vee$' indicated conjunction. In what follows, therefore, we assume that all translations meet that condition, taking it, in particular as an unstated part of Definitions 3.3.[20]

Each of the four modal logics considered Section 1 (following Makinson [40]), $L_{\mathbf{V}}$, $L_{\mathbf{I}}$, $L_{\mathbf{N}}$ and $L_{\mathbf{F}}$, is definitionally equivalent to CL. Translations establishing this are s_i, t_i, for $1 \leq i \leq 4$, where each s_i maps sentence letters to themselves and respects

[19] When translations are definitional our $\Box$-translations become the $\Box$-*definitional* translations of Humberstone [25], p. 266.

[20] It is possible that this assumption, or at least an assumption that t shows the nonmodal connectives the lesser respect of making, for example, $t(\alpha \wedge \beta)$ and $t(\alpha) \wedge t(\beta)$ logically equivalent, can be derived from the assumption that L is a modal logic in our sense and L' is CL, but we have been unable to establish whether this is so.

the non-modal connectives and $s_1(\Box\gamma) = \top$, $s_2(\Box\gamma) = s_2(\gamma)$, $s_3(\Box\gamma) = \neg s_3(\gamma)$, $s_4(\Box\gamma)=\bot$, and $t_1(\gamma) = t_2(\gamma) = t_3(\gamma) = t_4(\gamma) = \gamma$. Indeed, as the following theorem indicates, the Makinson four are essentially the only modal logics having this property.

THEOREM 3.5 *Suppose L is a modal logic in a language with one 1-ary modal connective $\Box$. Then L is definitionally equivalent to* CL *iff L is $L_{\mathbf{V}}$, $L_{\mathbf{I}}$, $L_{\mathbf{N}}$ or $L_{\mathbf{F}}$.*

Proof. The argument for the 'if' direction was given above. For the converse, suppose L is definitionally equivalent to CL via translations s and t. Then, since t and s are mutually inverse, $\Box p_0 \leftrightarrow t(s(\Box p_0)) \in L$. Since the only connectives present in the language of CL are non-modal and t respects the non-modal connectives and is variable-fixed, t must be the identity function. So $\Box p_0 \leftrightarrow s(\Box p_0) \in L$. Since s is definitional, $\Box p_0 \leftrightarrow \beta(p_0) \in L$ for some $\beta(p_0)$ in the language of CL whose only sentence letter is p_0. But then L must be one of the Makinson four. ■

Translations that might corroborate the suggestion beginning this section that AM is somehow tantamount to CL are not definitional. They comprise maps s and t from the language of AM to that of CL and back such that

- For all $i \geq 0, s(p_i) = p_{i+1}$,
- $s(\alpha \wedge \beta) = s(\alpha) \wedge s(\beta)$, and similarly for negation and other non-modal connectives,
- $s(\Box\alpha) = p_0$, and
- $t(p_0) = \Box p_0$,
- for all $i \geq 1, t(p_i) = p_{i-1}$,
- $t(\gamma \wedge \delta) = t(\gamma) \wedge t(\delta)$, and similarly for negation and other non-modal connectives.

The idea of using the first sentence letter in an official enumeration of them to play a distinguished role in translations and then shunting all the sentence letters of the source logic forward so that this causes no interference is familiar from translating from Johansson's minimal logic (with $\bot$ primitive, to be translated by the sentence letter singled out) into positive logic (i.e., intuitionistic logic in the $\{\wedge, \vee, \rightarrow\}$-fragment of intuitionistic logic).[21] In the case of minimal logic, there may be some inclination to resist the notion that $\bot$ doesn't merely behave like a sentence letter but actually is a (strangely written) sentence letter. This unease is reduced in the case of AM by the fact that, syntactically at least, $\Box$ is a 1-ary connective. It should not be so surprising to learn that $\Box\alpha$ is a kind of formula whose truth value is independent of α (and not just independent of α's truth-value).

[21] Alternatively, instead of being singled out once and for all, the sentence letter is taken to be one not occurring in a given α when defining the $\bot$-removing translation of α, as in Theorem C of Prawitz and Malmnäs [47].

The translations s and t above are neither variable-fixed nor narrowly compositional. They do, however, respect the nonmodal connectives (the background condition on all the translations in play in Definitions 3.3) and count as broadly compositional in the sense of Def. 3.3(v). (The template$^+$ formula $\beta(q_1)$ corresponding to $\Box$ here is just p_0, which contains zero occurrences of q_1. The fact that $q_1, \ldots, q_n$ need not all appear in the n-ary template$^+$ $\beta(q_1, \ldots, q_n)$ is a mere convenience. If it were required that they all appear and, say q_i were missing, we could replace $\beta(q_1, \ldots, q_n)$ by its equivalent $\beta(q_1, \ldots, q_n) \wedge (q_i \to q_i)$. In general, a broadly compositional translation replaces all the new "template" letters q_i in β – these correspond to the *parameters* introduced in Pelletier and Urquhart, [45], p. 263 – by formulas of the target language, so these translations really do take formulas of the source language to formulas of the target language. Any of the sentence letters p_i, in β's that were already in the target language, however, remain. Thus, a broadly compositional translation, even if it is variable-fixed, can map $\#(p_1, \ldots, p_n)$ to a formula containing sentence letters other than $p_1, \ldots, p_n$, as s maps $\Box p_1$ to p_0 here.[22]

Our next result uses the notion of compositional equivalence introduced in Definition 3.3(vi):

THEOREM 3.6 AM *is compositionally equivalent to* CL.

Proof. Let s and t be the translations above. For any formula γ in the language of CL, $s(t(\gamma)) = \gamma$, and so condition 4 on translational equivalence is satisfied. For any formula α in the language of AM, $t(s(\alpha))$ is the result of replacing certain subformula occurrences in α of the form $\Box\beta$ (in particular, those not contained in any larger such occurrences) with $\Box p_0$. Since AM is congruential, it is closed under replacement of equivalents. Since contains all instances of $\Box\alpha \leftrightarrow \Box\beta$, therefore, it contains $t(s(\alpha)) \leftrightarrow \alpha$, and so condition 3 is also satisfied. To check conditions 1 and 2 we make a detour through the semantics. It is convenient to write an assignment of truth values to sentence letters as a sequence $\langle \mathbf{a}(p_0), \mathbf{a}(p_1), \ldots \rangle$. (Notation here is as in the discussion following Corollary 2.7 above: $\mathbf{a}$ for assignments, "ξ" with or without subscripts, for truth-values.) Then we can associate with each pair-model $\mathcal{M} = \langle \xi, \langle \xi_1, \xi_2, \ldots \rangle \rangle$ the truth value assignment $\mathbf{a}_{\mathcal{M}} = \langle \xi, \xi_1, \xi_2, \ldots \rangle$ and with each truth-value assignment $\mathbf{a} = \langle \xi_1, \xi_2, \ldots \rangle$ the pair-model $\mathcal{M}(\mathbf{a}) = \langle \xi_1, \langle \xi_2, \ldots \rangle \rangle$. Note that the transitions from $\mathcal{M}$ to $\mathbf{a}$ and back are just a matter of deleting and adding the inner angle brackets, so $\mathbf{a}_{\mathcal{M}(\mathbf{a})} = \mathbf{a}$ and $\mathcal{M}(\mathbf{a}_{\mathcal{M}}) = \mathcal{M}$. A routine induction establishes that, for all formulas α in the language of AM, $\mathcal{M} \models \alpha$ iff $\mathbf{a}_{\mathcal{M}} \models s(\alpha)$ and for all formulas γ in the language of CL, $\mathbf{a} \models \gamma$ iff $\mathcal{M}(\mathbf{a}) \models t(\gamma)$. To show that condition 1 is satisfied suppose $\alpha \notin \mathsf{CL}$. Then there is some truth-value assignment $\mathbf{a}$ such that $\mathbf{a} \not\models \alpha$. By the induction above, $\mathcal{M}(\mathbf{a}) \not\models s(\gamma)$. Since all models verify the formulas in AM, this implies $s(\gamma) \notin \mathsf{AM}$, as required. Condition 2 is proved by a similar argument. ■

So we see that AM and the four Makinson logics are all compositionally equiva-

[22] The compositional translations of Def. 3.5 in Yang [59] are what we are calling broadly compositional translations.

lent to CL, whereas the latter four logics, and only those, meet the stronger condition of being definitionally equivalent to it.

It is natural to ask whether AM and the Makinson four are the only modal logics compositionally equivalent to CL. We approach this question indirectly. We show that there is a natural class of fifteen logics including these four, that bear to CL a weaker affinity relation that is of interest in its own right. As it turns out, these fifteen are the only modal logics that relate to CL in this way, and so any modal logics translationally equivalent to CL must be among them. In addition to AM we identify five others that are so equivalent. The question of whether any of the remaining four are translationally equivalent to CL is left open.

The class of logics in question were identified as the *extensional* logics in Humberstone ([16], [17]) and reidentified in a somewhat different way as the *prime* modal logics in Zolin ([60]). We briefly introduce this class of logics,[23] emphasizing a particular example for illustration. A modal logic is *extensional* if it contains all instances of $(\alpha \leftrightarrow \beta) \to (\Box\alpha \to \Box\beta)$. Note that this is a "horizontal" correlate of the congruentiality condition, and extensional logics are therefore congruential. Each logic of the Makinson four, in which $\Box$ can be interpreted by a particular 1-ary truth-function, is clearly extensional. But so also are all intersections of these logics. Consider, for example, the logic $L_{\mathbf{VI}} = L_{\mathbf{V}} \cap L_{\mathbf{I}}$. A formula is valid in this logic if it is verified by any Boolean valuation that treats $\Box$ consistently with either the Verum or the Identity truth functions. Thus $\Box$ (in the current presentation of this idea) is a hybrid of the two corresponding connectives, sharing the logical properties of both, just as in AM it was a hybrid of connectives corresponding to the Verum and Falsum truth-functions. Indeed, the extensional modal logics are precisely intersections of logics among the Makinson four.[24]

Our notation for the extensional logics subsumes the earlier notation for the Makinson four. The symbol 'L' subcripted by a word on the boldface alphabet $\mathbf{V}$, $\mathbf{F}$, $\mathbf{I}$ or $\mathbf{N}$ (the letters selected in that order) denotes a logic in which $\Box$ has the properties shared by the corresponding connectives. In other words the valid formulas are those verified by any Boolean valuation that treats $\Box$ consistently with any of the indicated truth functions. A Boolean valuation that interprets $\Box$ as $\mathbf{V}$, $\mathbf{F}$, $\mathbf{I}$ or $\mathbf{N}$ may be called a *Makinson valuation*, and so we may say that every extensional logic is determined by an appropriate class of Makinson valuations. In this notation AM would be called $L_{\mathbf{VF}}$. If we take $\mathbf{V}$, $\mathbf{F}$, $\mathbf{I}$ and $\mathbf{N}$ as one-letter words then the result of subscripting L by one of these letters again denotes one of the Makinson four, and we have $L_{\mathbf{VF}} = L_{\mathbf{V}} \cap L_{\mathbf{F}}$, $L_{\mathbf{FIN}} = L_{\mathbf{F}} \cap L_{\mathbf{I}} \cap L_{\mathbf{N}}$, etc. The first of these examples is a version of Corollary 2.8.[25] [26] In introducing his prime logics,

[23]Our exposition recapitulates Humberstone's *extensionality* (rather than Zolin's *primeness*), tailored to classically-based modal logic. For the less parochial general notion of extensionality (in sentence position), see note 35 below, and the text to which it is appended.

[24]The discussion in, for example, 3.21 of [22] can be readily adapted to show this.

[25]The one-letter labels were avoided in [16] and [17], where the words themselves denote logics, because the notation would then be ambiguous between logic and truth-function.

[26]Of course the first disjunct here could be written simply as $\Box p$, but writing it with biconditional disjuncts illustrates how we select one disjunct for each of the truth-functions being hybridized. Note that any of these disjunctions with more than one disjunct witnesses Halldén incompleteness, and

Zolin ([60]) provides alternative axiomatizations. Since these will be useful in what follows we reproduce them here. Listed below are the extensional logics (minus the Makinson four) written in our notation and the characteristic axioms given in [60], p. 866.

$L_{\mathbf{VF}}$	$\Box p \leftrightarrow \Box\bot$
$L_{\mathbf{VI}}$	$\Box p \leftrightarrow (p \vee \Box\bot)$
$L_{\mathbf{VN}}$	$\Box p \leftrightarrow (\neg p \vee \Box\top)$
$L_{\mathbf{FI}}$	$\Box p \leftrightarrow (p \wedge \Box\top)$
$L_{\mathbf{FN}}$	$\Box p \leftrightarrow (\neg p \wedge \Box\bot)$
$L_{\mathbf{IN}}$	$\Box p \leftrightarrow (p \leftrightarrow \Box\top)$
$L_{\mathbf{VFI}}$	$\Box p \leftrightarrow (\Box\bot \vee (\Box\top \wedge p))$
$L_{\mathbf{VFN}}$	$\Box p \leftrightarrow (\Box\top \vee (\Box\bot \wedge \neg p))$
$L_{\mathbf{VIN}}$	$\Box p \leftrightarrow ((p \leftrightarrow \Box\top) \vee (\neg p \leftrightarrow \Box\bot))$
$L_{\mathbf{FIN}}$	$\Box p \leftrightarrow ((p \leftrightarrow \Box\top) \wedge (\neg p \leftrightarrow \Box\bot))$
$L_{\mathbf{VFIN}}$	$\Box p \leftrightarrow ((p \wedge \Box\top) \vee (\neg p \wedge \Box\bot))$

Each axiom in the list contains only a single sentence letter. Our axiom schema $\Box\alpha \to \Box\beta$ for AM ($= L_{\mathbf{VF}}$), for example, is replaced by $\Box p \leftrightarrow \Box\bot$.[27] Furthermore each provides a logical equivalent to $\Box p$ that is a truth-functional combination of p and the "modal constants" $\Box\top$ and $\Box\bot$.

The extensional logic $L_{\mathbf{VI}}$ that we choose as our example is a well-known normal modal logic, called KT_c in essentially Brian Chellas's nomenclature (as in Humberstone [25]). It makes an appearance in two recent writings. Kracht [31] (p. 320), cites it as the smallest Fregean normal modal logic – where the term 'Fregean' is adapted from the usage illustrated in Czelakowski and Pigozzi [5] for consequence relations, to the case of logics as sets of formulas. Kocurek ([30] Proposition 20) proves that it is a sublogic of any normal modal logic compositionally embeddable into CL. Our Theorem 3.8 below shows that, as long as translations preserve non-modal connectives, it is itself so embeddable, and thus Kocurek's Proposition 20 establishes that it is the smallest normal logic to do so. Our Theorems 3.9 and 3.10 show that there are exactly 15 modal logics compositionally embeddable into CL. Of these only three, $L_{\mathbf{VI}}$, $L_{\mathbf{V}}$ and $L_{\mathbf{I}}$, are normal. As might be expected, all three are extensions of $L_{\mathbf{VI}}$. (Some caution is warranted here, since we are operating with logics as sets of formulas, and Kocurek, with logics as consequence relations.)

We now introduce the weaker affinity notion alluded to above. (Our terminology follows [30].)

Definitions 3.7 Logics L and L' are

(i) *intertranslatable* if L and L' are bi-directionally faithfully embeddable;

that the disjunction with all four disjuncts yields the same modal logic as does a representative instance – distinct sentence letters replacing distinct schematic letters – of the schema used to define (sentence position) extensionality here. This gives the logic of the 'completely generic' 1-ary extensional connective, as it is put in [16].

[27] Note that $\Box p \leftrightarrow \Box\top$ would do equally well here, and that, on either choice, we have here another example of the phenomenon investigated in [27].

(*ii*) *definitionally intertranslatable* if L and L' are intertranslatable by a pair of definitional translations;

(*iii*) *compositionally intertranslatable* if L and L' are intertranslatable by a pair of broadly compositional translations.

Relations among the various affinities between logics spelled out in Definitions 3.1(*iv*), 3.3(*iv*, *vi*) and 3.7 are shown in Figure 1. Solid arrows indicate that the target is strictly weaker than the source. (Bracketed numbers above the arrows are bibliographical references to proofs of that strictness, i.e., to proofs that the arrows go only in the direction indicated. See note 30 for details.) The dotted arrow labeled by a question mark indicates that the target is weaker than the source, but we don't know whether it is strictly weaker.

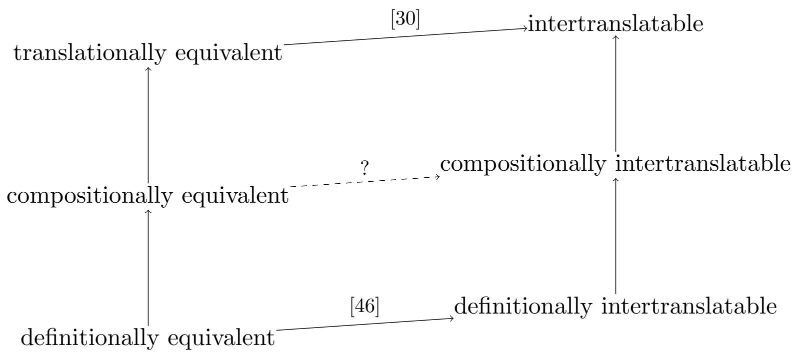

Figure 1: Affinities Between Logics

We now show that every extensional logic is translationally intertranslatable with CL– looking at our example $L_{\mathbf{VI}}$ in detail and sketching the argument for several others.

Theorem 3.8 *$L_{\mathbf{VI}}$ and CL are compositionally intertranslatable.*

Proof. The identity translation $t(\gamma) = \gamma$ provides a faithful embedding of CL into any consistent modal logic L, for the following reason. If $\gamma \in \mathsf{CL}$, then $t(\gamma) \in L$ because modal logics include all classical tautologies. If $\gamma \notin \mathsf{CL}$ then, for some substitution $\mathscr{s}$ replacing each sentence letter by $\top$ or by $\bot$, $\neg\mathscr{s}(\gamma) \in \mathsf{CL}$, so $\neg\mathscr{s}(\gamma) \in L$. The consistency of L thus implies that $\mathscr{s}(\gamma) \notin L$. Hence, since modal logics are closed under uniform substitution, $\gamma \notin L$. Taking L as $L_{\mathbf{VI}}$, then, to complete the proof it suffices to find a broadly compositional faithful embedding from $L_{\mathbf{VI}}$ into CL that respects the non-modal connectives. Let s be the (broadly compositional) translation with the following inductive definition:

$s(p_i) = p_{i+1}$,

$s(\neg\alpha) = \neg s(\alpha)$,

$s(\alpha \wedge \beta) = s(\alpha) \wedge s(\beta)$, and similarly for $\vee$ and $\rightarrow$,

$s(\Box\alpha) = p_0 \vee s(\alpha)$.

Since s respects the non-modal connectives, it maps tautologies to tautologies and satisfies the condition that $s(\beta)$ a tautological consequence of tautologies $s(\alpha_1), \ldots, s(\alpha_n)$ implies $s(\beta)$ also a tautology. Furthermore s maps the characteristic Zolin axiom for $L_{\mathbf{VI}}$ to $(p_0 \vee s(p)) \leftrightarrow (s(p) \vee (p_0 \vee \bot))$, which is a tautology. This establishes that s is an embedding of the kind required.

It remains only to show that s is faithful. Suppose $\alpha \notin L_{\mathbf{VI}}$. Then there is some Makinson valuation $u \in V_{\mathbf{V}} \cup V_{\mathbf{I}}$ such that $u(\alpha) = \mathrm{F}$. Suppose first that $u \in V_{\mathbf{V}}$. Let $\mathbf{a}$ be the truth value assignment that assigns T to p_0 and, for $i \geq 1$, $u(p_{i-1})$ to p_i. Then by formula induction it follows that, for all formulas β, $u(\beta) = \mathrm{T}$ iff $\mathbf{a} \models s(\beta)$, and so, in particular, $a \not\models s(\alpha)$. Now suppose that $u \notin V_v$. Then $u \in V_{\mathbf{I}}$. Let a be the same truth value assignment as before except that it now assigns F to p_0. Then again it follows by induction that $u(\beta) = \mathrm{T}$ iff $\mathbf{a} \models s(\beta)$. For the $\Box$ case here, we have the following chain of equivalences: $u(\Box\gamma) = \mathrm{T}$ iff $u(\gamma) = \mathrm{T}$ (because $u \in V_{\mathbf{VI}}$) iff $\mathbf{a} \models s(\gamma)$ (by induction hypothesis) iff $\mathbf{a} \models p_0 \vee s(\gamma)$ (since $\mathbf{a}(p_0) = \mathrm{F}$ iff $\mathbf{a} \models s(\Box\gamma)$. So again in this case we have $\mathbf{a} \not\models s(\alpha)$. ∎

Similar arguments apply to *every* intersection of one or more of the Makinson logics, so that Theorem 3.8 can be strengthened to the following result:

THEOREM 3.9 *Every extensional modal logic is compositionally intertranslatable with* CL.

To give some indication here of the kind of reasoning involved, it will help to consider what is going on in the proof of Theorem 3.8. The broadly compositional translation s in play had treated $\Box$ by means of the template$^+$ $\alpha = \alpha(q_1) = p_0 \vee q_1$. We wanted $\Box$ to behave as a hybrid of (connectives for) $\mathbf{V}$ and $\mathbf{I}$. But any Boolean valuation v for which $v(p_0) = \mathrm{T}$ satisfies $v(\alpha) = \mathrm{T}$ regardless of $v(q_1)$. So we could interpret $\Box$ as $\mathbf{V}$ on such valuations. If v is any other valuation, then $v(p_0) = \mathrm{F}$, and so $v(\alpha) = v(q)$. In that case $\Box$ could be interpreted as $\mathbf{I}$ on v. For brevity, let us abbreviate all this to:

$$\text{When } p_0 \mapsto T,\ \Box \mapsto \mathbf{V} \text{ and when } p_0 \mapsto \mathrm{F},\ \Box \mapsto \mathbf{I}.$$

The argument above handles the case of $L_{\mathbf{VI}}$. $L_{\mathbf{VF}}$, alias AM, is dealt with in Theorem 3.6 (giving the stronger compositional *equivalence* result). There remain four twofold hybrids to address ($L_{\mathbf{VN}}$, $L_{\mathbf{FI}}$, $L_{\mathbf{FN}}$ and $L_{\mathbf{IN}}$), as well as four threefold hybrids ($L_{\mathbf{VFI}}$, $L_{\mathbf{VFN}}$, $L_{\mathbf{VIN}}$, $L_{\mathbf{FIN}}$) and the one fourfold case ($L_{\mathbf{VFIN}}$). Templates$^+$ for the twofold cases can be recovered (if necessary) from Figure 3 in [26], reproduced here as Figure 4 in an Appendix to the present paper. Find the relevant hybridized pair in that table, note the corresponding 'Porte compound', and replace nullary constant Ω by p_0 and the variable p by q_1 (which we shall write simply as "q" for this discussion). The pairs are listed in that table as ordered pairs, but we can select either order for a particular hybrid. For example, the Ω compounds listed for $\langle \mathbf{I}, \mathbf{F} \rangle$ and $\langle \mathbf{F}, \mathbf{I} \rangle$ are respectively $\Omega \wedge p$ and $\neg\Omega \wedge p$, and we can take $\alpha(q)$ as $p_0 \wedge q$ or as

$\neg p_0 \wedge q$, obtaining equally workable templates$^+$ for present purposes.[28] For future reference, we display the templates$^+$ obtained from the Porte compounds, with the ones from those in our canonical order **V**, **F**, **I**, **N** listed first:

- For $L_{\mathbf{VN}}$ take $\alpha(q) = p_0 \vee \neg q$ or $\alpha(q) = \neg p_0 \vee \neg q$
- For $L_{\mathbf{FI}}$ take $\alpha(q) = \neg p_0 \wedge q$ or $\alpha(q) = p_0 \wedge q$
- For $L_{\mathbf{FN}}$ take $\alpha(q) = \neg p_0 \wedge \neg q$ or $\alpha(q) = p_0 \wedge \neg q$
- For $L_{\mathbf{IN}}$ take $\alpha(q) = p_0 \leftrightarrow q$ or $\alpha(q) = \neg p_0 \leftrightarrow q$

The remaining hybrids require templates$^+$ with *two* sentence letters, and so to accommodate this we set $s(p_i) = p_{i+2}$. Using the abbreviated notation introduced above, among the options, we have:[29]

- For $L_{\mathbf{VFI}}$, take $\alpha(q) = p_0 \wedge (p_1 \vee q)$. When $p_0, p_1 \mapsto$ T, $\Box \mapsto$ **V**; when $p_0 \mapsto$ T and $p_1 \mapsto$ F, $\Box \mapsto$ **I**; when $p_0 \mapsto$ F, $\Box \mapsto$ **F**.
- For $L_{\mathbf{VFN}}$, take $\alpha(q) = p_0 \vee \neg(p_1 \vee q)$. When $p_0 \mapsto$ T, $\Box \mapsto$ **V**; when $p_0 \mapsto$ F and $p_1 \mapsto$ T, $\Box \mapsto$ **F**; when $p_0, p_1 \mapsto$ F, $\Box \mapsto$ **N**.
- For $L_{\mathbf{VIN}}$, take $\alpha(q) = p_0 \vee (p_1 \leftrightarrow q)$. When $p_0 \mapsto$ T, $\Box \mapsto$ **V**; when $p_0 \mapsto$ F and $p_1 \mapsto$ T, $\Box \mapsto$ **I**; when $p_0, p_1 \mapsto$ F, $\Box \mapsto$ **N**.
- For $L_{\mathbf{FIN}}$, take $\alpha(q) = p_0 \wedge (p_1 \leftrightarrow q)$. When $p_0, p_1 \mapsto$ T, $\Box \mapsto$ **I**; when $p_0 \mapsto$ T and $p_1 \mapsto$ F, $\Box \mapsto$ **N**; when $p_0 \mapsto$ F, $\Box \mapsto$ **F**.
- For $L_{\mathbf{VFIN}}$, take $\alpha(q) = p_0 \leftrightarrow (p_1 \vee q)$. When $p_0, p_1 \mapsto$ T, $\Box \mapsto$ **V**; when $p_0 \mapsto$ T and $p_1 \mapsto$ F, $\Box \mapsto$ **I**; $p_0 \mapsto$ F and $p_1 \mapsto$ T, $\Box \mapsto$ **F**; when $p_0, p_1 \mapsto$ F, $\Box \mapsto$ **N**.

Thus, for example, the $\alpha(q)$ of this last case serves as an incarnation of the completely generic one-place extensional connective, mentioned in note 26.

The theorem below states that the extensional logics are the only modal logics that (within our framework) are compositionally intertranslatable with CL

[28] To understand the insignificance of order here it is useful to consider the characterization of logics that we are calling extensional by Łukasiewicz in [39]. Łukasiewicz noted that, for example, what we call $L_{\mathbf{IF}}$, is determined by a product matrix of the truth tables for the identity and falsum connectives, i.e., a matrix in which the new connective yields the value $\langle x', y' \rangle$ when applied to a formula taking value $\langle x, y \rangle$ iff $x' = x$ and $y' =$ F. Since the logic determined by the product of two matrices is the intersection of the logics determined by the factor matrices, the same logic would be determined by the product matrix of falsum and identity. Nevertheless, as Łukasiewicz observed, in languages with both connectives, the order is not insignificant. Using **IF** and **FI** for the two connectives, we have: $\mathbf{IF}\varphi \to \mathbf{FI}\varphi$ not valid (in the sense of always taking value $\langle \mathrm{T}, \mathrm{T} \rangle$) while $\mathbf{IF}\varphi \to \mathbf{IF}\varphi$ is valid and, conversely, $(\mathbf{IF}\varphi \vee \mathbf{FI}\varphi) \leftrightarrow \varphi$ valid while, $(\mathbf{IF}\varphi \vee \mathbf{IF}\varphi) \leftrightarrow \varphi$ is not. For more on this see [26].

[29] There are in general several non-equivalent options for each case. For example in place of the first one listed here, $\alpha(q) = p_0 \wedge (p_1 \vee q_1)$, we could equally well have cited the result of interchanging conjunction and disjunction in this formula.

THEOREM 3.10 *Suppose a modal logic L is faithfully embedded into* CL *by a broadly compositional translation that respects the non-modal connectives. Then L is extensional.*

Proof. Let t be a translation of the sort supposed. Then $t(\Box p_0) = \beta(t(p_0))$ and $t(\Box p_1) = \beta(t(p_1))$, where $\beta(q_1)$ is a formula in the language of CL supplemented by a sentence letter q_1 and, for $i = 0, 1$, $\beta(t(p_i))$ is the result of replacing all (zero or more) occurrences of q_1 in $\beta(q_1)$ by $t(p_i)$. It follows that $(t(p_0) \leftrightarrow t(p_1)) \to t(\Box p_1) \leftrightarrow t(\Box p_1)) \in \mathsf{CL}$. Since t respects $\leftrightarrow$ and $\to$, $t((p_0 \leftrightarrow p_1) \to (\Box p_0 \leftrightarrow \Box p_1)) \in \mathsf{CL}$. Since t is faithful, $(p_0 \leftrightarrow p_1) \to (\Box p_0 \leftrightarrow \Box p_1) \in L$, which implies that L is extensional, as was to be shown. ■

We asked above whether AM and the Makinson four are the only logics that are compositionally equivalent to CL. This led to the observation that all the extensional logics and only are at least compositionally intertranslatable with CL. But the translations witnessing this are not mutually inverse. For example, if s and t are the translations from $L_{\mathbf{VI}}$ to CL and back, then $t(s(\Box p_0))$ is $p_0 \vee p_1$, which is not $L_{\mathbf{VI}}$-equivalent to $\Box p_0$. It is known that intertranslatability does not imply translational equivalence and that definitional intertranslatability does not imply definitional equivalence.[30] We don't know the answer to the corresponding question for compositional intertranslatability and compositional equivalence. But at least in this case there are other broadly compositional translations between the extensional logics and CL that *are* mutually inverse and so the hybrid extensional logics are compositionally equivalent to CL. The following strengthening of Theorem 3.8 illustrates this.

THEOREM 3.11 *$L_{\mathbf{VI}}$ is compositionally equivalent to* CL*.*

To prove this, the following result giving sufficient conditions for compositional equivalence will be convenient.[31]

LEMMA 3.12 *If s faithfully embeds L into L' and $s(t(\beta)) \leftrightarrow \beta \in L'$ then L is compositionally equivalent to L' via s, t.*

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[30] See Kocurek [30], Proposition 6, for the first fact and Pelletier and Urquhart [46], Theorem 2.1, for the second. The result that Pelletier and Urquhart use to establish that their example logics are not equivalent (*viz.*, Theorem 4.3 of [45]) requires that translations be what we might call "loosely definitional". Connectives are translated via templates but a sentence letter p_i may be translated by an arbitrary formula in which no sentence letters other than p_i occur. Thus Theorem 2.1 does not establish that either the top or middle arrow of Figure 1 is one-directional. The translations used for the example logics are definitional, however, and they are $\Box$-translations, so the proof of Theorem 2.1 does establish that the bottom arrow is one-directional. Analogous notions and results have been reported for first order theories. See, for example, Corcoran [4], Andréka *et al.* [1] and Barrett and Halvorson [2], [36], [28], and chapter 7 of [13]. Not surprisingly, parallels between notions of definitional equivalence (intertranslatability, etc.), as applied to *logics*, and their namesakes as applied to *theories* have found their way into recent debate over 'exceptionalism about logic'; see Dewar [6], and references there cited.

[31] This is Proposition 2.2.4 in Kocurek [29] p. 64, adapted to the present framework.

Proof. Suppose L is faithfully embedded into L' via s, and $s(t(\beta)) \leftrightarrow \beta \in L'$. To prove that t embeds L' into L note that $\beta \in L'$ iff $s(t(\beta)) \in L'$, and, since s is a faithful embedding, this happens iff $t(\beta) \in L$. It remains only to prove that $t(s(\alpha)) \leftrightarrow \alpha \in L$. By the hypothesis of the lemma, $s(t(s(\alpha))) \leftrightarrow s(\alpha) \in L'$. Since s respects the non-modal connectives $s(t(s(\alpha)) \leftrightarrow \alpha) \in L'$. Since s is faithful, $t(s(\alpha)) \leftrightarrow \alpha \in L$ as required. ∎

Proof of theorem. Let s be as in the proof of Theorem 3.8 and let t be the (broadly compositional) translation in the other direction given by:

$t(p_0) = \Box\bot$,

$t(p_i) = p_{i-1}$ for $i > 0$,

$t(\neg\alpha) = \neg t(\alpha)$,

$t(\alpha \wedge \beta) = t(\alpha) \wedge t(\beta)$, and similarly for $\vee$ and $\rightarrow$.

We know from the proof of Theorem 3.8 that s is a faithful embedding of $L_{\mathbf{VI}}$ into CL. By the previous lemma, then, it is sufficient to prove that $s(t(\alpha)) \leftrightarrow \alpha \in \mathsf{CL}$. We do so by induction on the construction of α. Because s and t respect the non-modal connectives and CL satisfies replacement of equivalents, we need only consider the base case. If $i = 0$ then $s(t(p_i)) = s(\Box\bot) = (p_0 \vee \bot)$, and so $s(t(p_i)) \leftrightarrow p_i \in L_{\mathbf{VI}}$. If $i > 0$ then $s(t(p_i)) = p_i$ and $s(t(p_i)) \leftrightarrow p_i$ follows directly. ∎

To adapt the reasoning used in the proof of Theorem 3.11 to the extensional logics characterizing the remaining "binary" hybrids, $L_{\mathbf{VN}}$, $L_{\mathbf{FI}}$, $L_{\mathbf{FN}}$ and $L_{\mathbf{IN}}$, one can take s to be an appropriate translation from those described in the argument for Theorem 3.9 and adjust the value of $t(p_0)$ so that $s(t(p_0)) \leftrightarrow p_0 \in \mathsf{CL}$ is provable. (For the remaining sentence letters we just let t decrease subscripts by one so that $s(t(p_i)) = p_i$.)

THEOREM 3.13 *$L_{\mathbf{VN}}$, $L_{\mathbf{FI}}$, $L_{\mathbf{FN}}$ and $L_{\mathbf{IN}}$, are each compositionally equivalent to* CL

Proof. Following the strategy just outlined, for $L_{\mathbf{VN}}$ let $s(\Box\alpha) = p_0 \vee \neg s(\alpha)$ and $t(p_0) = \Box\top$; for $L_{\mathbf{FI}}$ let $s(\Box\alpha) = p_0 \wedge \alpha$ and $t(p_0) = \Box\top$; for $L_{\mathbf{FN}}$ let $s(\Box\alpha) = p_0 \wedge \neg s(\alpha)$ and $t(p_0) = \Box\bot$; for $L_{\mathbf{IN}}$ let $s(\Box\alpha) = p_0 \leftrightarrow s(\alpha)$ and $t(p_0) = \Box\top$. In the first case $s(t(p_0)) = p_0 \vee \neg\top$; in the second case$s(t(p_0)) = p_0 \wedge \top$; in the third case $s(t(p_0)) = p_0 \vee \neg\top$; in the fourth case $s(t(p_0)) = p_0 \leftrightarrow \top$. In each case, $s(t(p_0)$ is tautologically equivalent to p_0, as desired. ∎

One might think that this strategy could be extended to cover the logics for the three- and four-place hybrids. One would look for values for both $t(p_0)$ and $t(p_1)$ that would make it possible to show that both $s(t(p_0)) \leftrightarrow p_0$ and $s(t(p_1)) \leftrightarrow p_1$ were in CL. For example, in the case of $L_{\mathbf{VFI}}$ it would be natural to take $s(\Box\alpha) = p_0 \wedge (p_1 \vee s(\alpha))$ and $t(p_0) = \Box\top$. We would then have $s(t(p_0)) = p_0 \wedge (p_1 \vee \top)$, which is equivalent to

p_0 as required. But there appears to be no value that we could assign to $t(p_1)$ that would satisfy $s(t(p_1)) \leftrightarrow p_1 \in \mathsf{CL}$, and We have been unable to determine whether or not the logics for the three- and four-place hybrids are compositionally equivalent to CL.

The results of this section are summarized in Figure 2. The smaller rectangles represent individual logics, the Makinson logics comprising the adjoining rectangles in the lower portion of the diagram, and the binary intersections of these shown above them. The larger nested rectangles containing these represent classes of logics. Affinity to CL is represented by shading: darker shading indicating closer affinity. The asterisk in the legend indicates that the question of whether all extensional logics are compositionally equivalent to CL is not settled here. Thus the logics in each rectangle bear at least the affinity to CL indicated in the legend and, with the possible exception of those extensional logics that are neither Makinson logics nor binary intersections thereof, they bear no closer affinity.[32]

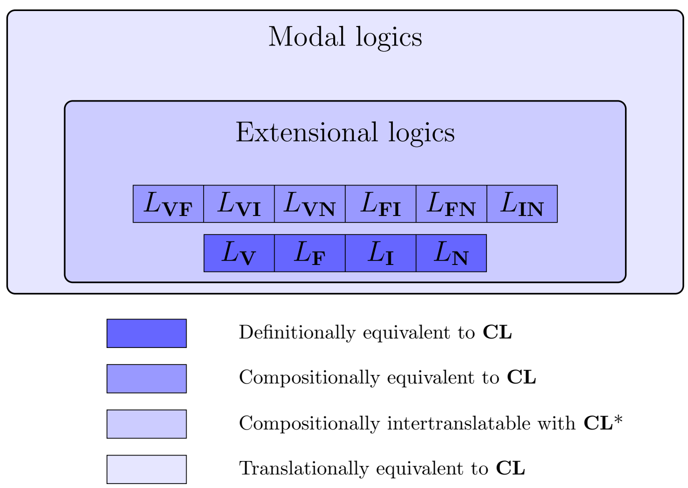

Figure 2: Affinities Between Modal Logics and CL

4 Hybridizing the Unit and Zero Operators in a More General Setting

In previous sections, modal logics were understood as as extensions of classical propositional logic in a language expanded by the addition of the 1-ary connective $\Box$. It emerged in Section 2 that such logics are characterized by the presence of the

[32]To be more precise, *affinities* here are equivalence relations between logics, each associated with a particular condition. One affinity is *closer* than another if its associated condition is logically stronger. One should not assume that affinities like those appearing in Figure 1 are always linearly ordered by closeness, and much less so when these are supplemented by forms of translational isomorphism like that mentioned in note 40. However, the four affinities listed in the legend of Figure 2 *are* linearly ordered by closeness when restricted to the class of modal logics considered here, and the first three of these are generally so ordered.

theorem $\Box\alpha \to \Box\beta$, and that are exactly three of them. The weakest, which can be axiomatized by that formula alone, we called AM. The other two, which we called $L_{\mathbf{V}}$ and $L_{\mathbf{F}}$, are just the unit and zero logics from Makinson's well-known result about the extensions of modal logics, and AM is their intersection. In section 3 it was seen that AM is just one member of the family of 16 *extensional* logics, each an intersection of one or more of Makinson's four logics, and all of which bear close affinities to classical propositional logic. In this section we wish to focus attention more closely on just $\Box$ itself without any presumptions as to how it might interact with other connectives. For that purpose the most convenient logical framework is not one seeing logics as sets of formulas meeting appropriate conditions, or even as consequence relations, but as generalized (or 'multiple-conclusion') consequence relations.[33] There are natural analogs of both the amphitone property and the extensional logics in these more general settings, and the analogs of $L_{\mathbf{VI}}$ again have the amphitone property. Unlike $L_{\mathbf{VI}}$, itself, however, they are not many-valued logics, nor are they the smallest logics with the amphitone property. Furthermore, as will be seen, their amphitone status is not so special: every extensional logic in the more general settings is amphitone. We conclude the section by asking about conditions under which the results of the previous section *do* carry over to the more general frameworks.

To maintain the parallel with our previous framework, we restrict attention to congruential relations, which we now take to mean that $\alpha \dashv\vdash \beta$ only if $\Box\alpha \dashv\vdash \Box\beta$, whether $\vdash$ is a consequence relation or a generalized one. For convenience we will often refer to consequence relations and generalized consequence relations as crs and gcrs, respectively, and for clarity we will use the notation '$\Vdash$' in place of '$\vdash$' for gcrs. Since we take $\Box$ as the sole connective in the language, the idea of $\Box\alpha$'s provably implying $\Box\beta$ is, in the absence of any implicational connective, represented by its being the case that $\Box\alpha \Vdash \Box\beta$. A cr is *monotone* if $\alpha \vdash \beta$ implies $\Box\alpha \vdash \Box\beta$; *antitone* if $\alpha \vdash \beta$ implies $\Box\beta \vdash \Box\alpha$, and amphitone if both monotone and antitone. For gcrs the notions are understood similarly.

A treatment of gcrs in the spare language of interest to us can be retrieved from Humberstone [17] and in particular we can extract from the lattice of Figure 1 in that work (p. 347) the sublattice relevant to our concerns.[34] The original lattice shows the inclusion relations among all 26 gcrs $\Vdash$ on a language with $\Box$ as its sole connective where $\Box$ is *extensional* according to $\Vdash$ in the sense that for all α, β, we have $\alpha, \beta, \Box\alpha \Vdash \Box\beta$ and $\Box\alpha \Vdash \Box\beta, \alpha, \beta$. These conditions require, respectively, that $\Box$ treats all truths alike and that $\Box$ treats all falsehoods alike.[35] Together,

[33] See note 4 for background.

[34] In the original diagram (and discussion), $\vdash$ is written rather than $\Vdash$, schematic letters are $A, B, \ldots$ rather than α, β, and $\Box$ appears as $\#$.

[35] That is, every valuation v that is consistent with $\Vdash$ in the sense that it never assigns T to every formula in Γ and F to every formula in Δ when $\Gamma \Vdash \Delta$ satisfies the conditions (i) that if $v(\alpha) = v(\beta) =$T then $v(\Box\alpha) = v(\Box\beta)$ and (ii) that if $v(\alpha) = v(\beta) =$ F then $v(\Box\alpha) = v(\Box\beta)$. Thus on every valuation consistent with $\Vdash$, $\Box$ is associated in the obvious sense with a 1-ary truth-function – though not necessarily the same truth-function for different $\Vdash$-consistent valuations, thereby distinguishing extensionality (in sentence position) from truth-functionality, as it was put in [16]; a discussion removing an element of unclarity in this contrast is provided by Chapter 3 of

they amount to a formulation of extensionality as defined here, now purged of any dependence on connectives other than $\Box$ itself, and hence of any assumptions as to the logical behavior of such connectives. We may now check that all extensional gcrs are congruential.[36] Suppose $\Vdash$ is extensional and $\alpha \Vdash \beta$ and $\beta \Vdash \alpha$. Extensionality implies $\alpha, \beta, \Box\alpha \Vdash \Box\beta$. Since $\alpha \Vdash \beta$, we have $\alpha, \Box\alpha \Vdash \Box\beta$. Similarly, $\Box\alpha \Vdash \Box\beta, \alpha, \beta$ and $\beta \Vdash \alpha$ imply that $\Box\alpha \Vdash \Box\beta, \alpha$. Together, this implies $\Box\alpha \Vdash \Box\beta$, and so $\Vdash$ is indeed congruential. Note also that the extensionality conditions are satisfied by any $\Vdash$ for which we have $\Box\alpha \Vdash \Box\beta$ unrestrictedly. We depict the relevant sublattice in Figure 3.

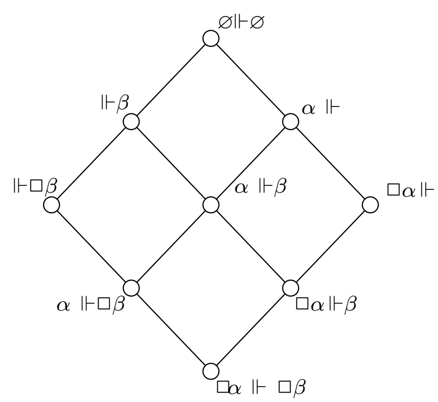

Figure 3: Some Extensional Gcrs

A node in Figure 3 represents the smallest gcr satisfying the condition written beside that node (for all formulas α, β); the empty set is indicated by a gap except in the case of the top node, which represents the inconsistent gcr for which we have $\Gamma \vdash \Delta$ for all sets Γ, Δ of formulas. One node is lower than another if the logic represented by the first is a sublogic (a subrelation, that is) of that represented by the second. Thus there are nine logics in this restricted language, all extending

[22]. There the truth-functionality of a connective is taken to be relative to a class of valuations, but the extensionality of a connective is relative to a cr or to a gcr, and is a purely syntactic matter. So a semantic analog of extensionality is to be provided – "pseudo-truth-functionality" – as well as a syntactic analog of truth-functionality, in the terminology "is fully determined (according to $\vdash$ or $\Vdash$)," so that like can be compared with like. The details of the comparisons depend on whether it is crs or gcrs that are at issue, however, and for consequence relations it does not coincide with extensionality in the sense of Czelakowski and Pigozzi [5], p. 56, which counts the familiar (primitive or defined) connectives of intuitionistic logic as extensional according to $\vdash_{\mathsf{IL}}$, although they do not treat falsehoods (= non-truths: only bivalent valuations are at issue) in the same way. (See Observation 3.23.2 in [22] for this, and more generally, §3.3.) [5]'s extensionality corresponds to the *weak extensionality* of [22].

[36] This fact can perhaps be appreciated more easily by considering the semantic gloss on the extensionality conditions given above. For $\alpha \Vdash \beta$ and $\beta \Vdash \alpha$ together indicate that α and β have the same truth value on any $\Vdash$-consistent valuation, and so the two extensionality conditions imply that $\Box\alpha \Vdash \Box\beta$.

the the logic $L_{\mathbf{VF}}$ from Section 3, now conceived of as a gcr and denoted by $\Vdash_{\mathbf{VF}}$, sitting at the bottom of the lattice. (Boldface letters as before.) As in Section 3, the label reflects the idea that this logic treats $\Box$ as a hybrid of the constant true and constant false connectives. The central node represents a logic in which every formula is a consequence of every other, and, since gcrs are closed under *Weakening* (i.e., $\Gamma \Vdash \Delta$ implies $\Gamma' \Vdash \Delta'$ for $\Gamma \subseteq \Gamma'$ and $\Delta \subseteq \Delta'$), every non-empty set is a consequence of every non-empty set. The definition of generalized consequence does not require that it follow from this that the empty set implies or is implied by, any set, and so this logic has the three distinct extensions shown. If we regard the logic represented by the central node and the three logics properly extending it as trivial[37] all the logics extending that represented by the central node as trivial or essentially the same as that logic, then the **VF**-logic still has four non-trivial proper extensions. Descriptions of these logics (in fact, all 26 logics) in terms of valuational semantics can be found in [17].[38] As with the formula logic $L_{\mathbf{VF}}$ (alias AM) of the preceding section, $\Vdash_{\mathbf{VF}}$ treats $\Box$ as a hybrid of (i.e., something possessing precisely the shared unconditional $\Vdash$-properties of) $\Box$ as a constant true and $\Box$ as a constant false connective. That is $\Vdash_{\mathbf{VF}} = \Vdash_{\mathbf{V}} \cap \Vdash_{\mathbf{F}}$. This is represented in Figure 3 by the fact that the meet (intersection) of the leftmost and rightmost logics in the diagram is the bottom logic there. It is also the meet of the two logics on the level immediately above it, which are in the picture because we include gcrs with which the valuations that verify and falsify all formulas are consistent (in the sense of note 35). (These valuations are called v_T and v_F in [17].) They would be excluded if attention were restricted to what are assertive logics in Segerberg [50], p. 40, renaming the earlier 'regular' logics of Kuhn [33].[39]

In the current setting, the combination of being monotone with being antitone amounts to $\Vdash$'s satisfying the condition that for any α, β if $\alpha \Vdash \beta$ then $\Box\alpha \Vdash \Box\beta$ and $\Box\beta \Vdash \Box\alpha$ – a condition clearly satisfied by $\Vdash_{\mathbf{VF}}$. All the gcrs extending $\Vdash_{\mathbf{VF}}$ (and so appearing in Figure 3 if they do not also expand the language) are both monotone and antitone in this sense. The converse, however, no longer obtains.

To illustrate this, let us begin by considering gcrs $\Vdash$ (crs $\vdash$) with the property that $\alpha \Vdash \beta$ implies $\beta \Vdash \alpha$ (resp., $\alpha \vdash \beta$ implies $\beta \vdash \alpha$) for all α, β in the language of $\Vdash$ (resp., of $\vdash$). We may call such relations *pseudo-symmetric*–their restriction to singleton sets is symmetric. Any such $\Vdash$ or $\vdash$ which is either monotone or antitone is evidently amphitone.

Example 4.1 Let $\Vdash_{\mathbf{N}}$ be the least gcr on the language with sole connective (1-ary) $\Box$, to satisfy both $\alpha, \Box\alpha \Vdash$ and $\Vdash \alpha, \Box\alpha$ for all α. This is easily seen to be

[37] These are analogs for gcrs, of consequence relations sometimes called *inconsistent* and *almost inconsistent*. See, for example, [48].

[38] The discussion in [17], understanding connectives as individuated by their logical powers, speaks of Figure 3 as charting nine different *1-ary connectives* – or, rather, Figure 1 of that paper as charting 26 such connectives – though some might regard this as rather unnatural description in the trivial cases for which the logic is characterized by a condition not mentioning the connective itself.

[39] The 'regular' terminology clashed with an unrelated usage in modal logic for a certain class of logics between the class of all monotone logics and all normal modal logics, Kuhn [33] in fact uses the term 'consequence relations' rather than 'logics', and means by it what are here called generalized consequence relations.

antitone and pseudo-symmetric. It is Makinson's 'complement logic' transposed to the current setting. Since $\alpha \Vdash_{\mathbf{N}} \beta$ implies $\Box\beta \Vdash_{\mathbf{N}} \Box\alpha$, $\Vdash_{\mathbf{N}}$ is congruential. $\Vdash_{\mathbf{N}}$ is pseudo-symmetric in virtue of being symmetric, something that can't happen in the case of consequence relations. Accordingly, $\Vdash_{\mathbf{N}}$ is congruential and amphitone, despite not satisfying the **VF** condition $\Box\alpha \Vdash \Box\beta$. ◀

Similar remarks apply to $\Vdash_{\mathbf{I}}$–the least such gcr to satisfy both $\alpha \Vdash \Box\alpha$ and $\Box\alpha \Vdash \alpha$ (which is the gcr version of $L_{\mathbf{I}}$). Since $\Vdash_{\mathbf{V}}$ and $\Vdash_{\mathbf{F}}$ are amphitone and the amphitone property for gcrs is preserved under intersections, this implies that, when logics are construed as gcrs, every extensional logic in the language with no connectives other than $\Box$ is amphitone. There is an even simpler example worth considering, that of the smallest gcr on the language with sole connective $\Box$. In this case $\Gamma \Vdash \Delta$ iff $\Gamma \cap \Delta \neq \varnothing$, immediately rendering $\Vdash$ both congruential and amphitone since $\alpha \Vdash \beta$ just when α and β are the same formula, in which case so are $\Box\alpha$ and $\Box\beta$. This last example illustrates another way in which the amphitone property on logics as gcrs differs from that on logics as sets of formulas. In the latter case it is preserved when the logic is extended, not so in the former: we can have amphitone $\Vdash$ and $\Vdash^{+} \supseteq \Vdash$, with $\Vdash^{+}$ not amphitone. Variants of these examples arise also for the corresponding consequence relations. If $\vdash_{\mathbf{N}}$ is taken as the restriction of $\vdash_{\mathsf{CL}}$ to the language in which negation is the sole connective (which could be rewritten as $\Box$ for complete parity with the foregoing, if desired) then we have a pseudo-symmetric consequence relation which, being antitone, is also amphitone, without extending $\vdash_{\mathbf{VF}}$. And corresponding to the simple minimal case with gcrs, if we take the consequence relation on the $\Box$-only language, in which formulas are consequences of precisely the sets of which they are elements, and we again have the $\alpha \vdash \beta \ \Rightarrow\ \alpha = \beta$ situation.[40] So, although the amphitone property coincides with the provability of $\Box\alpha \to \Box\beta$ when logics are construed as sets of formulas, it does not coincide with the corresponding condition $\Box\alpha \vdash \Box\beta$ (characterizing $\vdash_{\mathbf{VF}}$) for consequence relations.

We can better understand the divergence between logics as sets of formulas and logics as crs or gcrs with regard to the amphitone property by recalling the proof

[40]The situation is quite different when, as in earlier sections, logics are construed as sets of formulas in a language with, say, $\neg, \wedge, \vee, \to$ and $\leftrightarrow$ interpreted classically. In this case the minimal logic is Makinson's L_0 from [41] (with $\neg, \wedge, \vee, \to$ and $\leftrightarrow$ as primitives), or Kuhn's 'operator logic' from [34] (restricted to a single 1-ary operator), i.e., it is the set of formulas true under every Boolean valuation, which is the set of substitution instances of tautologies. This logic is not congruential (much less amphitone), containing, for example, formulas of the form $(\alpha \vee \beta) \leftrightarrow (\beta \vee \alpha)$ but not those of the form $\Box(\alpha \vee \beta) \to \Box(\beta \vee \alpha)$. Although this logic is outside the class considered in this paper, it may be interesting to note its relation to CL. The most obvious translations from L_0 to CL would divide the sentence letters into two countable subsets (say p_i with i odd and p_i with i even), map the sentence letters of L_0 1-1 to the former, the $\Box$-governed truth-functional constituents ('Boolean atoms') of L_0 1-1 to the latter, and the non-modal combinations of these constituents to the corresponding non-modal combinations of their translations. (Note the contrast here with AM, where every truth-functional constituent was mapped to the *same* sentence letter.) This is a faithful embedding that is not either narrowly or broadly compositional. It is, however, 1-1 and onto, and so L_0 and CL are, in Kocurek's terminology, *isomorphic*. (See [30], Definition 2.11.) As Kocurek shows, if L and L' are isomorphic via t, then they are translationally equivalent via t and t^{-1}.

of Theorem 2.1. There we appealed to the presence in our language of $\top$ and $\bot$ (as either 0-ary connectives or formula abbreviations) as well as $\to$. But the identity of those particular expressions (or at least the first two of them) was not essential to the proof. The same argument would go through as long as any formulas α and β are both *uplinked* (i.e., $\alpha \to \gamma$ and $\beta \to \gamma$ are theorems for some formula γ) and *downlinked* (i.e., $\delta \to \alpha$ and $\delta \to \beta$ are theorems for some formula δ). Thus, if we extend the notion of a congruential logic to encompass any logics in which occurrences of α can be replaced by β whenever $\alpha \leftrightarrow \beta$ is provable (this having been put in terms of the "replacement of equivalents" in the course of the proof of Theorem 3.11), then the Lindenbaum algebra of a congruential logic, formulas' being uplinked (downlinked) corresponds to their equivalence classes' having an upper bound (resp. a lower bound).[41] Since $\delta \to \alpha$ and $\beta \to \gamma$ are in the logic, the antitone property implies that $\Box\alpha \to \Box\gamma$ and $\Box\delta \to \Box\beta$ are as well. But a familiar property of the conditional ensures that $\delta \to \gamma$ is a theorem and so the monotone property implies $\Box\delta \to \Box\gamma$ is as well. Indeed, an even simpler version of the argument will go through as long as any two formulas are *directly linked*, i.e., they are either uplinked or downlinked. If, for example α and β are uplinked by γ, then the monotone property ensures that $\Box\alpha \to \Box\gamma$ and the antitone property ensures that $\Box\gamma \to \Box\beta$. If α and β are downlinked the argument is similar. These considerations show that Theorem 2.1 would obtain in other logics, even those in languages lacking some of the connectives of CL. The notion of linkage defined above and the definitions of the monotone and antitone properties require the presence of $\to$, and the proof uses the principle that $\alpha \to \gamma$ is an element of the logic if $\alpha \to \beta$ and $\beta \to \gamma$ are, but no other connectives or logical principles are required. Even in a logic as weak as implicational *BCK*,[42] every pair of formulas is directly linked, since both are uplinked by any provable formula (e.g., $p \to p$). In the present context, however, where there may be no connectives other than $\Box$, things are somewhat different. The condition that $\alpha \to \beta$ is a theorem is now replaced by $\alpha \vdash \beta$ or $\alpha \Vdash \beta$, and the notions of linkage are revised accordingly. In this case there may be pairs of formulas with no direct links, and, as we have seen, the analog of Theorem 2.1 may fail. Indeed, taking the consequence relation case, if we have either of $\vdash \alpha \to \beta$ and $\alpha \vdash \beta$ without the other, then we may have to live with a corresponding bifurcation between two linkage notions. For example in the pure implicational *BCI* logic[43] we again have $p \to p$ provable. Any associated consequence relation $\vdash$ therefore satisfies $\alpha \vdash p \to p$ for all α, and so any two formulas are '$\vdash$-uplinked'. But now

[41] Note the contrast with two somewhat similarly defined notions in [22], p. 1319*ff.* and elsewhere, such as the references given at the top of p. 1336 there: formulas α, β are there defined to be *head-linked* if for some formulas $\alpha_0, \beta_0, \gamma$, they are respectively equivalent to $\alpha_0 \to \gamma$ and $\beta_0 \to \gamma$, and *tail-linked* if for some formulas $\alpha_0, \beta_0, \delta$, they are respectively equivalent to $\delta \to \alpha_0$ and $\delta \to \beta_0$. (The linking formula γ or δ appears here at the 'head' end or 'tail' end of the "$\to$".) Head-linked formulas in IL exhibit strikingly CL-like behavior; for example α and β are head-linked in IL iff they enjoy there the Peircean equivalences (1): of α with $(\alpha \to \beta) \to \alpha$, and (2): of β with $(\beta \to \alpha) \to \beta$.

[42] See Došen [7], which usefully describes the history of this logic and its weaker relative *BCI* mentioned below.

[43] See Došen [7] for background; for further information see the index entries in [22] under '*BCI* logic' (and '*BCK* logic)'.

we no longer have $\vdash \alpha \to (p \to p)$, which fails when α is not itself provable in BCI (and sometimes when it is). From this, it is not hard to see that p and q are not '$\to$-uplinked' in BCI.

Setting aside such $\to$-dependent notions of linkage, it would be interesting to see if some variation on the theme of 'every pair of formulas is directly linked' might turn out to be a necessary as well as a sufficient condition for any simultaneously monotone and antitone $\Box$ to be constant in a cr or gcr with no assumptions about the underlying language. For ease of exposition, let's consider gcrs. One plausible variation would be this: define *being linked* according to $\Vdash$ as the ancestral of the relation of being directly linked according to $\Vdash$. The arguments above show that for any formulas α and β, α directly linked to β according to $\Vdash$ implies $\Box\alpha \Vdash \Box\beta$. It follows that every pair of formulas being linked is sufficient for $\Box\alpha \Vdash \Box\beta$ to characterize the amphitone property. Now suppose $\Vdash$ is amphitone and, for all α and β, $\Box\alpha \Vdash \Box\beta$. Does it follow that every pair of formulas is linked according to $\Vdash$? More generally, does $\Box\alpha \Vdash \Box\beta$ imply that α and β are linked?

5 Conclusion

David Makinson ([40]) draws attention to the four modal logics determined by modal algebras obtained by adding a 1-ary operator $*$ to the Boolean algebra with elements in $\{0,1\}$: $L_{\mathbf{V}}$ (for which $*0 = *1 = 1$), $L_{\mathbf{F}}$ (for which $*0 = *1 = 0$), $L_{\mathbf{I}}$ (for which $*x = x$), and $L_{\mathbf{N}}$ (for which $*x$ is the complement of x). He shows that every monotone modal logic is a sublogic of $L_{\mathbf{V}}$, $L_{\mathbf{F}}$ or $L_{\mathbf{I}}$, and that every antitone modal logic is a sublogic of $L_{\mathbf{V}}$, $L_{\mathbf{F}}$ or $L_{\mathbf{N}}$. Here we extend Makinson's results to show that every *amphitone* modal logic (i.e., every modal logic that is both monotone and antitone) is a sublogic of either $L_{\mathbf{F}}$ or $L_{\mathbf{I}}$. We show further there are exactly three such logics: $L_{\mathbf{F}}$, $L_{\mathbf{I}}$ and the minimal amphitone logic, AM, that can be axiomatized by adding the axiom $\Box\alpha \to \Box\beta$ to a standard axiomatization of the minimal congruential modal logic E. Makinson's four logics, and only those, are definitionally equivalent (in a sense described here) to the classical propositional logic CL. Intersections of any two of these logics bear to CL the slightly weaker relation of compositional equivalence. Intersections of three or more of Makinson's logics, bear to CL the still weaker relation of compositional intertranslatability. Furthermore Makinson logics and their (binary, ternary or quaternary) intersections are the only modal logics to do so. Most of our discussion, like Makinson's, is conducted within a framework that regards modal logics as sets of formulas in a language equipped with a truth-functionally complete set of nonmodal connectives. In a more general framework, where a logic is regarded as a substitution-invariant gcr, $\Vdash$, between sets of formulas and sets of formulas and no assumptions are made about the nonmodal connectives present, a natural analog of AM is $\Vdash_{\mathbf{VF}}$, represented by the bottom node in Figure 3. $\Vdash_{\mathbf{VF}}$ is the smallest gcr relation satisfying $\Box\alpha \Vdash \Box\beta$ (or, more pedantically, $\{\Box\alpha\}\Vdash\{\Box\beta\}$) for all α, β. This gcr and all its extensions are both monotone and antitone. In this setting, however, there are other amphitone logics. Indeed (as Example 4.1 illustrates) the amphitone property is no longer preserved when the logic extended, and (as the last case of that example illustrates) $\Vdash_{\mathbf{VF}}$ is not even a smallest amphitone gcr. For this reason we do not follow the example set by the notation "$\vdash_{\mathsf{AM}}$" from Section 2. (See the text to which note 11 is appended.) This divergence occurs not so much because of the contrast between consequence relations and generalized consequence relation as because the lack of additional logical vocabulary in the present setting. If a logic in this framework has the property that every pair of formulas is directly linked, in the sense that $\alpha \Vdash \beta$ implies that either $\alpha \Vdash \gamma$ and $\beta \Vdash \gamma$ for some γ, or $\gamma \Vdash \alpha$ and $\gamma \Vdash \beta$ for some γ, then the original findings are restored: being amphitone coincides with extending $\Vdash_{\mathbf{VF}}$.

In the course of our discussion two questions were raised that remain open:

- Suppose $\Vdash$ is amphitone and $\Vdash \supseteq \Vdash_{\mathbf{VF}}$. Does it follow that every two formulas are linked in $\Vdash$ (where being linked is the ancestral of being directly linked)?
- Are any logics other than the Makinson four and their binary intersections compositionally equivalent to CL? In view of Theorem 3.10, only the five remaining extensional logics are live candidates. A negative answer would settle the question of whether compositional intertranslatability implies compositional equivalence.

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Appendix

For the reader's convenience in connection with the discussion after Theorem 3.9, we make available as Figure 4, appearing as Figure 3 in [26] (with the more explicit caption "Porte–Vakarelov Constant-induced Operators"). As explained there, to keep the layout compact, overlining is used for negation, (connectives for) the constant true and constant false binary truth-functions are written as Ⓣ and Ⓕ, with the the projections to the first and second coordinate as ① and ②, respectively.

Ω compound	Hybridized pair	Ω compound	Hybridized pair
$\Omega \wedge p$	$\langle \mathbf{I}, \mathbf{F} \rangle$	$\Omega \vee p$	$\langle \mathbf{V}, \mathbf{I} \rangle$
$\Omega \wedge \overline{p}$	$\langle \mathbf{N}, \mathbf{F} \rangle$	$\Omega \vee \overline{p}$	$\langle \mathbf{V}, \mathbf{N} \rangle$
$\overline{\Omega} \wedge p$	$\langle \mathbf{F}, \mathbf{I} \rangle$	$\overline{\Omega} \vee p$	$\langle \mathbf{I}, \mathbf{V} \rangle$
$\overline{\Omega} \wedge \overline{p}$	$\langle \mathbf{F}, \mathbf{N} \rangle$	$\overline{\Omega} \vee \overline{p}$	$\langle \mathbf{N}, \mathbf{V} \rangle$
$\Omega \leftrightarrow p$	$\langle \mathbf{I}, \mathbf{N} \rangle$	Ω ① p	$\langle \mathbf{V}, \mathbf{F} \rangle$
$\overline{\Omega} \leftrightarrow p$	$\langle \mathbf{N}, \mathbf{I} \rangle$	$\overline{\Omega}$ ① p	$\langle \mathbf{F}, \mathbf{V} \rangle$
Ω Ⓣ p	$\langle \mathbf{V}, \mathbf{V} \rangle$	Ω ② p	$\langle \mathbf{I}, \mathbf{I} \rangle$
Ω Ⓕ p	$\langle \mathbf{F}, \mathbf{F} \rangle$	Ω ② $\overline{p}$	$\langle \mathbf{N}, \mathbf{N} \rangle$

Figure 4: Porte–style Constant-induced Operators